Chapter 2 – Factors: How Time and Interest Affect Money

Standard Factor Notation

- General Form: \( (X/Y, i, n) \) or \( (X/Y) \)
  - “X given Y – factor”, tabulated for different \( i, n \)
  - \( X = \text{variable to be calculated} \)
  - \( Y = \text{known variable} \)
  - \( i = \text{interest rate per period} \)
  - \( n = \text{number of interest periods} \)

Single-Payment Factors (F/P and P/F)

Recall that \( F = P(1 + i)^n \)

Given \( P \), to find \( F \), \( (1+i)^n \) is the F/P conversion factor

\[
F = P \left(1 + \frac{i}{100}\right)^n
\]

\( (F/P, i, n) \) is tabulated for different \( i \) and \( n \)

FV(I%,n,P) in Excel

Example C1:
If $1,000 were deposited in a bank savings account, how much would be in the account in two years if the bank paid 4% interest compounded annually?

Solution: \( P = $1,000, n = 2, i = 4\% \), \( F = ? \)

\[
F = P(1+i)^n = $1,000 \left(1 + \frac{0.04}{100}\right)^2 = $1,081.60, \text{ or}
\]

The table factor \( (F/P, 4\%, 2) \) is 1.0816 (p. 710), therefore

\[
F = P \left(1 + \frac{0.04}{100}\right)^2 = $1,000 \times 1.0816 = $1,081.60
\]
Single-Payment Factors (F/P and P/F)

*(P/F)*: Single Payment Present Worth Factor

Find \( P \) (\( i \) and \( n \) are also given)

\[ F \text{ given} \]

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Single-Payment Factors (F/P and P/F)

Since \( F = P(1 + i)^n \)

\[ P = F / (1 + i)^n \]

\[ 1/(1+i)^n \text{ is the P/F conversion factor} \]

\[ P = F (P/F) \]

*(P/F, i, n)* is tabulated for different \( i \) and \( n \)

\[ PV(i\%, n, F) \text{ in Excel} \]

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Example C2:
If you wished to have $1,082 in a savings account at the end of two years and 4% interest was paid annually, how much should you put into the savings account now?

Solution: \( P = ?, \ n = 2, \ i = 4\%, \ F = $1,082 \)

\[ P = F / (1+i)^n = $1,082 / (1 + 0.04)^2 = $1,000.37, \text{ or} \]

\[ P = F (P/F, 4\%, 2) = $1,000.37 \]

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F/P and P/F Factors

- Simplest factors
- Similar ideas are used in developing more factors for more complex cash flow types

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Uniform-Series Factors

Uniform Series Present Worth Factor \((P/A)\)

\[ P = \] \[ \begin{array}{cccc}
0 & 1 & \ldots & n-1 & n \\
A & A & \ldots & A & A \\
\end{array} \]

- Objective: Find \( P \), given \( A \)

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Uniform-Series Factors

- Uniform-Series Present Worth Factor \((P/A)\)
- Using tables: \( P = A (P/A, i, n) \)
- Using Excel: \( P = PV(i\%, n, A) \)

\[ P = \frac{A(1+i)^n-1}{(1+i)^n i} \]
Uniform-Series Factors

Capital Recovery Factor (A/P)

- Objective: Find A, given P.

\[
P = A \left( \frac{1}{(1+i)^n} \right)
\]

Using tables: \( A = P \left( \frac{A}{P}, i, n \right) \)
Using Excel: \( A = PMT(i\%, n, P) \)

Uniform-Series Factors

Uniform-Series Compound Amount Factor (F/A)

- Objective: Find F, given A

\[
F = \frac{A}{1+i} \left( \frac{1}{1+i} \right)^{n-1}
\]

Using tables: \( F = P \left( F/A, i, n \right) \)
Using Excel: \( F = FV(i\%, n, A) \)

Uniform-Series Factors

Sinking Fund Factor (A/F)

- Objective: Find A, given F.

\[
A = F \left( \frac{1}{(1+i)^n} \right)
\]

Using tables: \( A = F \left( A/F, i, n \right) \)
Using tables: \( A = PMT(i\%, n, F) \)
Standard Factor Notation

<table>
<thead>
<tr>
<th>To Find</th>
<th>Given</th>
<th>Factor</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>F</td>
<td>(P/F, i, n)</td>
<td>P = F*(P/F, i, n)</td>
</tr>
<tr>
<td>F</td>
<td>P</td>
<td>(F/P, i, n)</td>
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</tr>
</tbody>
</table>

Practice deriving factor formulas using geometric sum identity

Example C3:

Tom deposits $500 in his saving account at the end of each year for 24 years and the bank pays 6% interest per year, compounded yearly. What are the present worth and future worth of this yearly investment.

Solution:

\[ P = \frac{500[(1+0.06)^24-1]}{(1+0.06)^24(0.06)} = \$ 6275.18 \] (or use P/A factor from the table = 500 * 12.5504 = $6275.20)

F = ?

Solution (cont.):

\[ F = \frac{500[(1+0.06)^24-1]}{0.06} = \$ 25,407.79 \]

Example C4:

A proximity sensor attached to the tip of an endoscope could reduce risks during eye surgery by alerting surgeons to the location of critical retinal tissue. If a certain eye surgeon expects that by using this technology, he will avoid lawsuits of $0.5 and $1.25 million 2 and 5 years from now, respectively, how much could he afford to spend now if his out-of-pocket costs for the lawsuits would be only 10% of the total amount of each suit? Use an interest rate of 8% per year.

Solution:

\[ P = 50,000 (P/F,8\%,2) + 125,000 (P/F,8\%,5) = \$ 127,940 \]
Example C5:
A company which uses austenitic nickel-chromium alloys to manufacture resistance heating wire is considering a new annealing-drawing process to reduce costs. If the new process will cost $1.75 million dollars now, how much must be saved each year to recover the investment in 10 years at an interest rate of 12% per year?

Solution
\[ A = 1.75 \text{ million} \times (A/P, 12\%, 10) = 309,715 \]

Uniform-Series Factors

Example C6:
What initial investment is needed in order that an income of $400 per year can be made for 5 years?

Assume an annual interest rate of 15%.

Solution:
\[ A = 400, \quad i = 15\%, \quad n = 5, \quad P = ? \]
\[ P = 400 \times (1 + 0.15)^5 - 1 / (1 + 0.15)^5(0.15) = 1,341 \]

What is the corresponding value of F?

Compound Interest Factor Tables

Factor values are listed in tables 1-29 at the end of the text (pgs. 702 – 730)

- i: 0.25% - 50%
- n: 1 - 480

Example C7: Billy wishes to save enough money to buy a new car. He will place a sum in a savings account today and again in three years in anticipation of spending $15,000 in five years. What should the amount be? Take i = 12%.

Solution:
\[ 15,000 = S(1+i)^5 + S(1+i)^3 = S(1.12)^5 + S(1.12)^3 \]
\[ = S(3.016742) \]
\[ S = 15,000 / 3.016742 = 4,972 \]

Alternatively,
\[ [S(1+i)^5 + S(1+i)^3] = 15,000 \]
\[ S(1+i)^5 + S(1+i)^3 = 15,000 \]
\[ S = 4,972 \]
Alternatively

Using Tables

\[ 15,000 = S(P|F, 12\%, 5) + S(P|F, 12\%, 2) \]
\[ = S(1.7623) + S(1.2544) \]
\[ = S(3.0167) \]
\[ S = \frac{15,000}{3.0167} = 4,972 \]

Example C8:

Bill wants to make deposits each year for five years to buy the $15,000 car. His first payment will be one year from today. How big must the deposits be if interest is 12% per year?

Solution:

a) Using Tables

\[ $15,000 = A(F|A, 12\%, 5) = A(6.3528) \]
\[ A = \frac{15,000}{6.3528} = 2,361 \]

Compound Interest Factor Tables

b) Solution: Using Formula

\[ A = \frac{\frac{15,000 \times 0.12}{(1+0.12)^5 - 1}}{1+0.12} \]
\[ = \frac{15,000 \times 0.12}{(1.12)^5 - 1} = 2,361 \]

c) Using Spreadsheet PMT function

\[ A = \text{PMT}(12\%, 5, 0, 15000) = 2,361.15 \]

Interpolation in Interest Tables

- Read Section 2.4

Arithmetic Gradient Factors

- An arithmetic gradient is a cash flow series that either increases or decreases by a constant amount each period.
- The base amount \( A_1 \) \((A)\) is the uniform-series amount that begins in period 1 and extends through period \( n \).
- Starting with the second period, each payment is greater (or smaller) than the previous one by a constant amount referred to as the arithmetic gradient \( G \).
- \( G \) can be positive or negative.
Arithmetic Gradient Factors

**Base Amount**

\[ A_1, A_2, A_3, \ldots, A_n \]

**Gradient (the base is ignored)**

\[ G, (n-1)G, (n-2)G, \ldots, 2G, 3G, \ldots, (n-1)G, A_1 \]

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Chapter 2 Factors: How Time and Interest Affect Money

### Case 1: Find \( P \) given \( G \)

The total present worth \( P \) can be found as the sum of the present values of the series of base payments \( P_A \) and the present worth of the series of gradient payments \( P_G \):

\[ P = P_A + P_G \]

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### Arithmetic Gradient Factors

We will study the following three cases:

- **Case 1**: Find \( P \) given \( G \)
- **Case 2**: Find \( F \) given \( G \)
- **Case 3**: Find \( A \) given \( G \)

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### Case 2: Find \( F \) given \( G \)

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### Case 3: Find \( A \) given \( G \)

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### The present worth of the base payment \( A \) for each period 1 through \( n \) is:

\[ P_A = A_1 \left( \frac{P}{A_i}, i, n \right) \]

---

### The present worth of the gradient payments \( G \) for each period 2 through \( n \) is:

\[ P_G = \frac{G}{i(1+i)^n} \left[ \frac{(1+i)^n - 1}{i} - n \right] = G \left( \frac{P}{G}, i, n \right) \]
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Arithmetic Gradient Factors

Case 2: Find $F$ given $G$ and $A_1$

Similarly, $F = F_A + F_G$

Since

$$P_G = \frac{G}{i(1+i)^n} \left[\frac{(1+i)^n-1}{i}\right] - n$$

and $F = P(1+i)^n$

$$F_G = \frac{G}{i} \left[\frac{(1+i)^n-1}{i}\right] - n$$

Also,

$$F_A = A\left[\frac{(1+i)^n-1}{i}\right] = A_1 \left(F/A, i, n\right)$$

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Arithmetic Gradient Factors

Case 3: Find $A$ given $G$ and $A_1$

$$A_T = A_1 + AG$$

Since and

$$F = G \left(A/G, i, n\right)$$

$$A_g = \frac{G}{i} \left[\frac{(1+i)^n-1}{i}\right] - n\left[\frac{i}{(1+i)^n-1}\right] = G\left[1 - n\frac{i}{(1+i)^n-1}\right]$$

$$= G \left(A/G, i, n\right)$$

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Arithmetic Gradient Factors

Example C9:

$100$ is deposited in a saving account one year from today, $200$ two years from now, $300$ three years from now, ..., $1000$ ten years from now. What is the value of the account in ten years if interest is $7\%$.

Solution:

Example 2.8 Cash Flow Diagram

(*1,200)

(*1,000)

(*800)

(*600)

(*400)

(*200)

$0$

0123456789 1 0

Period

Cash Flow

Example 2.8: Cash Flow Diagram

Uniform Gradient

($200)$

($400)$

($600)$

($800)$

($1,000)$

($1,2000)$

0123456789 1 0

Period

Cash Flow

Solution:

$$A_1 = -100, G = -100, n = 10$$

$$F_T = F_A + F_G$$

$$= -100 \left(F|A, 7\%, 10\right) +$$

$$+ \left(-100 \left(P|G, 7\%, 10\right) \left(F|P, 7\%, 10\right)\right)$$

$$= -100 \left(13.8164 + (27.7156)(1.9672)\right)$$

$$= -6833.85$$

What are the values for $P_T$ and $A_T$?
Arithmetic Gradient Factors
Solution:

\[ P_T = F_T \left( \frac{P}{F}, 7\%, 10 \right) = -3473.64 \]
\[ A_T = F_T \left( \frac{A}{F}, 7\%, 10 \right) = -494.63 \]

Geometric Gradient Series Factors
- A cash flow series that either increases or decreases from period to period by a constant percentage.
- \( g \): constant rate of change, in decimal form, by which amounts increase or decrease from one period to the next (it's the geometric gradient).
- Initial amount \( A_1 \) in year 1, year 2 cash flow is \( A_1(1+g), \ldots, \) year \( n \) cash flow is \( A_1(1+g)^{n-1} \)

\[ P_g \] is the total present worth for the entire cash flow series

a) \( g \neq i, \quad P_g = A_1 \left[ 1 - \left( \frac{1 + g}{1 + i} \right)^n \right] \]

b) \( g = i, \quad P_g = nA_1/(1+i) \)

Geometric Gradient Series Factors

Example C10:
A pick-up big wheel modification costs $8000 and is expected to last 6 years with a $1300 salvage value. The maintenance cost is expected to be $1700 for the first year, increasing 11% per year thereafter. Determine the total equivalent present worth of modification if the interest rate is 8% per year.

Solution: \( A_1 = 1700, \quad g = 0.11, \quad i = 0.08, \quad n = 6 \)

Initial cost of $8,000 and annual maintenance costs are negative cash flows; salvage value of $1,300 is a positive cash flow (revenue).

\[ P_T = -8,000 - P_g + 1,300\left( \frac{P}{F}, 8\%, 6 \right) \]

\[ = -8,000 -1,700[1-(1.11/1.08)^6]/(-0.03) + 1,300(0.6302) \]

\[ = -8,000 -1,700(5.9559) + 819.26 \]

\[ = -$17,305.85 \]