Rational Inefficiency: A Nash-Cournot Oligopolistic Market Equilibrium

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Abstract

The standard assumption in the efficiency literature that firms desire to produce on the production frontier may not hold in an oligopolistic market where the production decisions of all firms will determine the market price, i.e. an increase in a firm’s output level leads to a lower market clearing price and potentially-lower profits. This paper models both the production possibility set and the inverse demand function and identifies a Nash-Cournot equilibrium and improvement targets which may not be on the production frontier. This behavior is referred to as rational inefficiency because the firm reduces its productivity levels in order to increase profits. For a general multiple input/output production process which allows a firm to adjust its output levels and variable input levels, the existence and the uniqueness of the Nash-Cournot equilibrium is proven. The relationship between the benchmark frontier, scale properties and allocative efficiency is discussed. When changes in quantity have a significant influence on price, we observe more benchmark production plans on the increasing returns to scale portion of the frontier. Additionally, a direction for improvement towards the allocatively efficient production plan is estimated, thus providing a solution to the direction selection issue in a directional distance analysis.

Keywords: Cournot-Nash Equilibrium, Oligopolistic Market, Allocative Efficiency, Nonparametric Frontiers
1. Introduction

Standard productivity and efficiency analysis assumes perfectly competitive markets and exogenous prices (Cherchye et al., 2002). Basic microeconomic theory states that firms operating in less than perfectly competitive markets can reduce production levels and increase a product’s market price when they face a downward sloping demand curve. Considering a monopolistically competitive market, Johnson and Ruggiero (2011) demonstrate from a revenue efficiency perspective that a firm that increases output to become technically efficient may actually reduce its overall profits by increasing the market quantity, which in turn reduces the market price. Figures 1 and 2 illustrate the endogenous prices of an oligopolistic market for a single product produced using a single input. The production frontier\(^1\) in figure 1 represents technically efficient production. Firms A and B would like to expand their output levels\(^2\) to increase their productivity, yet increasing the output levels will lead to a change in the market output quantity from Y to Y' (shown in figure 2) and the market price will fall from P to P'. This change in price may reduce the profits of both firms. Thus, the standard assumption in the efficiency literature that all firms desire to produce on the production frontier may not hold in an oligopolistic market (Cherchye et al., 2002; Johnson and Ruggiero, 2011). A firm is said to be rationally inefficient when it tries to maximize revenues or profits, or alternatively, minimize costs by intentionally operating at lower productivity levels. This paper considers an oligopolistic market to estimate a firm’s target production plans that may not be on the production frontier, but that maximize revenues or profits, or alternatively, minimize costs. The set of all firms’ benchmark production plans is a Nash-Cournot oligopolistic market equilibrium.

Most of the efficiency and productivity literature adapts the work developed by Farrell (1957) and articulated by Leibenstein (1966) as concepts X-efficiency which assumes that deviations from a production frontier are due to managerial inefficiency, lack of motivation, and lack of knowledge (Leibenstein and Maital, 1994). However, Stigler (1976) argues that firms and individuals are rational, meaning that what is observed as inefficiency is actually the difference

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\(^1\) A production function is commonly defined as the maximum set of output(s) that can be produced with a given set of inputs. Thus we will use the terms production function and production frontier interchangeably as is commonly done in the productivity and efficiency literature (Fried et al., 2008).

\(^2\) Firms will either expand their outputs, contract their inputs, or both, depending on the cost/price structure of inputs/outputs and adjustment costs associated with changing input levels. For now we will assume input adjustment costs are very large and consider only output adjustment consistent with an output oriented efficiency analysis in the efficiency literature, (Fare and Primont, 1995). This assumption is relaxed in section 4.
between individual employees of the firm maximizing their individual value functions and the firm’s value function. Following Stigler, Bogetoft and Hougaard (2003) suggest that if the inefficiency is due to lack of motivation, performance may be improved by redesigning the incentive structure to stimulate employees to save inputs and expand outputs.

Figure 1 Economic efficiency and production frontier

![Graph showing production function with firms A and B at different points on the frontier.]

Figure 2 Change in supply and equilibrium price

![Graph showing supply and demand curves with price and output quantities.]

In the case that these incentives or enforcement cost are higher than the cost of the inefficiency, it is rational for the firm to allow inefficient operations. Modeling the firm’s intention as maximizing profits and the employees’ intention as maximizing slack, the Bogetoft and
Hougaard show the trade-offs between the consumption of different types of slack. Alternatively, Wibe (2008) considers a firm that does not scrap older equipment as new models become available. His dynamic production model demonstrates that a considerable proportion of cross-sectional technical inefficiency can be rational economic behavior in terms of capital acquisition, i.e. he shows the role of capital (fixed inputs) in rational inefficiency.

In this paper we propose that rational inefficiency may in fact be a result of endogenous prices and the effect of output production on price – and profits. Cournot (1838), the first to consider endogenous prices, assumes a homogeneous product with an inverse demand function known to all firms which then independently select output levels; in this market characterized by imperfect competition, price is treated as an endogenous parameter. Nash (1950, 1951) considers more general non-cooperative games and defines a self-countering n-tuple as an equilibrium point in n-person games, i.e. for an equilibrium point, no firm can increase its objective function by unilaterally changing the quantity or price to any other feasible point. These games are consistent with the oligopolies described by Cournot where each firm maximizes its own profits and the output decisions affect the price faced by all firms. Rosen (1965) proves that a finite non-cooperative game always has at least one equilibrium point when the strategy space of each player is restricted to a simplex and the payoff functions are a bilinear function of the strategies. Further, for a constrained n-person game, he proves the existence and uniqueness of an equilibrium point with a strictly concave payoff function. A systematic discussion applying equilibrium concepts to economic systems is developed in Arrow and Debreu (1954). Discussing different classes of non-cooperative games, Milgrom and Roberts (1990) argue that all have identical bounds on the rationalizable strategies. In this paper we consider production strategies bounded by the production possibility set.

Murphy et al. (1982) introduce a mathematical programming approach for finding Nash equilibria in oligopolistic markets. They show that if the revenue function is concave and the cost function is convex and continuously differentiable, and the inverse demand function is strictly decreasing and continuously differentiable, then a Nash equilibrium solution exists if and only if a solution to the Karush–Kuhn–Tucker (KKT) conditions exist. Based on their study, Harker (1984) presents a variational inequality (VI) approach to find a Nash equilibrium using an
iterative procedure called the diagonalization algorithm. Bonanno (1990) gives a comprehensive survey on equilibrium theory with imperfect competition.

We use a variational inequality approach to identify Nash equilibria when production is limited by an endogenously estimated production frontier. We focus on an oligopolistic market with endogenous prices and firms maximizing profits. We identify a Nash equilibrium in which each firm cannot improve its profit by changing production levels within the production possibility set. We find that, contrary to previous productivity and efficiency studies, under certain conditions some firms choose not to produce on the production frontier, and we interpret the behavior as rational inefficiency (choosing to be less productive in order to increase profits).

The remainder of this paper is organized as follows. Section 2 shows the equivalence between a Nash equilibrium and the two approaches, variational inequalities and the complementarity problem, when production is restricted to the production possibility set. Section 3 examines revenue maximization with fixed input levels. Both a single output case and a multiple output case are presented. Section 4 introduces a generalized profit model in which a firm maximizes profits by adjusting both input and output levels. The existence and uniqueness of a Nash equilibrium identified through the complementarity problem is proven, and the relationship between the benchmark frontier and scale properties is discussed. Based on moving towards allocative efficient production, the direction for improvement used in the directional distance function is identified using the results of the Nash equilibrium analysis. Section 5 presents our conclusions.

2. Extending Approaches to Identify a Nash Equilibrium in Production Possibility Set

This section considers a general profit function and a production function with multiple inputs and multiple outputs, describes the conditions under which a Nash equilibrium solution exists, and how to identify it. We discuss the equivalence between the general concept of a Nash equilibrium and a set of variational inequalities and the complementary problem (CP) when production is limited to the production possibility set.

Let \( x \in R^I_+ \) denote the inputs and \( y \in R^Q_+ \) denote the outputs of a production system.
$Q = 1$ in the single output case. The production possibility set defined as $T = \{(x, y): \text{x can produce y}\}$ is estimated by a piece-wise linear convex function enveloping all observations (Farrell, 1957; Boles, 1967; Afriat, 1972; Charnes et al., 1978). The boundary of the production possibility set is referred to as the production frontier. For firm $k$, $X_{k1}$ is the $i^{th}$ input resource, $Y_{kq}$ is the amount of the $q^{th}$ production output, and $\lambda_k$ is the multiplier to construct convex combinations. Equation (1) uses a dataset characterizing firms to estimate the smallest set that imposes monotonicity and convexity on the production function, the boundary of the production possibility set $\bar{T}$.

$$\bar{T} = \left\{ (x, y) \left| \begin{array}{l}
\sum_k \lambda_k Y_{kq} \geq Y_q \quad \forall q; \\
\sum_k \lambda_k X_{ki} \leq X_i \quad \forall i; \\
\sum_k \lambda_k = 1; \\
\lambda_k \geq 0 \quad \forall k;
\end{array} \right. \right\}$$

(1)

To identify a Nash equilibrium, the generalized profit function should be concave, the inverse demand function should be nonincreasing and continuously differentiable, and the inverse supply function should be nondecreasing and continuously differentiable. The variational inequality approach and mixed complementary problem (MCP) are proven to be alternative methods to calculating a Nash equilibrium within the production possibility set.

To discuss the equilibria in oligopolistic markets characterized by imperfect competition, we define a Nash equilibrium problem (NEP) with respect to production possibility set as:

**Definition 1**: Let $K$ be a finite number of players, $\theta_k$ a utility (profit) function, $T_k$ a strategy set (production possibility set) for player $k = 1, ..., K$, and $(x_k, y_k) = (x_{k1}, ..., x_{ki}, y_{k1}, ..., y_{kQ}) \in T_k$ an observed production vector; then a vector $(x^*, y^*) = ((x_1^*, y_1^*), (x_2^*, y_2^*), ..., (x_K^*, y_K^*)) \in T_1 \times T_2 \times ... \times T_K = T$ is called a Nash equilibrium and is a solution to the NEP if

$$\theta(x^*, y^*) \geq \theta(x_k, x_{(-k)}^*, y_k, y_{(-k)}^*), \forall (x_k, y_k) \in T_k,$$

where $x_{(-r)} = (x_1^*, ..., x_{k-1}^*, x_{k+1}^*, ..., x_K^*)$ and $y_{(-r)} = (y_1^*, ..., y_{k-1}^*, y_{k+1}^*, ..., y_K^*)$ holds for all $k = 1, ..., K$.

Considering an NEP, Facchinei and Pang (2003) build a rigorous relationship among Nash equilibria, a set of variational inequalities (VI), and the complementarity problem (CP). We
restate their results for the scenario in which a production function bounds the production possibility set, and consider a profit function as a specific utility function.

**Lemma 1**: Let output levels be decision variables denoted by $y_{rq}$ as output $q$ of firm $r$ and $y_{rq} \geq 0$; further, let input levels be decision variables denoted by $x_{ri}$ as input $i$ of firm $r$, $x_{ri} \geq 0$, and $(x_{ri}, y_{rq}) \in \bar{T}$. Then define $P_q^r(y_{rq})y_{rq}$ as a concave function of $y_{rq}$ and assume that either the inverse demand function $P_q^r(y_{rq})$ is a non-increasing or a convex function of $y_{rq}$. Thus, for each $Y_{(-r)q} > 0$, where $Y_{(-r)q} = \sum_{k \neq r} y_{kq}$, $P_q^r(y_{rq} + Y_{(-r)q})y_{rq}$ is a concave function of $y_{rq}$ for $y_{rq} \geq 0$. Similarly, let $P_i^X(x_{ri} + X_{(-r)i})x_{ri}$ be a convex function of $x_{ri}$ for $x_{ri} \geq 0$, where $X_{(-r)i} = \sum_{k \neq r} x_{ki}$ and $P_i^X(x_{ri})$ is an inverse supply function. Further, if either $P_q^r(y_{rq})$ is strictly decreasing or is strictly convex, then $P_q^r(y_{rq} + Y_{(-r)q})y_{rq}$ is a strictly concave function on the nonnegative $y_{rq} \geq 0$ and $\sum_q P_q^r(y_{rq} + Y_{(-r)q})y_{rq} - \sum_i P_i^X(x_{ri} + X_{(-r)i})x_{ri}$ is a concave function on $(x_{ri}, y_{rq}) \in \bar{T}$.

Lemma 1 is important because it states that a global Nash equilibrium solution exists when the profit function $\sum_q P_q^r(y_{rq} + Y_{(-r)q})y_{rq} - \sum_i P_i^X(x_{ri} + X_{(-r)i})x_{ri}$ is concave and production is limited to a convex production possibility set. Generally, input markets are assumed to be competitive, in which case $P_i^X(x_{ri} + X_{(-r)i})$ is a constant, but in this case the lemma and the related results shown in theorems 1 and 2 and proven in section 4 still hold.

Gabay and Moulin (1980) propose that a Nash equilibrium will satisfy a set of VI. We reformulate the VI set with respect to the production possibility set:

**Theorem 1**: If the profit function of firm $r$, $\theta_r(x_{ri}, y_{rq}) = \sum_q P_q^r(y_{rq})y_{rq} - \sum_i P_i^X(x_{ri})x_{ri}$ is concave with respect to $(x_{ri}, y_{rq})$ and continuously differentiable, where $Y_q = \sum_k y_{kq}$ and $X_i = \sum_k x_{ki}$, then $(x^*, y^*) \in \bar{T}$ is a Nash-Cournot oligopolistic market equilibrium if and only if it satisfies the set of VI $\langle F((x^*, y^*)), (x, y) - (x^*, y^*) \rangle \geq 0$, $\forall (x, y) \in \bar{T}$. That is, $\sum_k F_k((x^*, y^*))(x_k, y_k) - (x^*_k, y^*_k) \geq 0$ $\forall (x_k, y_k) \in \bar{T}$, where
Consider an oligopoly with $K$ firms, an inverse demand function $P^Y(\cdot)$ that is strictly decreasing and continuously differentiable in $y$, and an inverse supply function $P^X(\cdot)$ that is strictly increasing and continuously differentiable in $x$. Since lemma 1 shows that the profit function $\theta_k(x_k, y_k)$ is concave and $x_k, y_k \geq 0$, then $(x^*, y^*) = ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_K^*, y_K^*))$ is a Nash equilibrium solution if and only if

$$\nabla_{x_k} \theta_k(x^*, y^*) \leq 0 \text{ and } \nabla_{y_k} \theta_k(x^*, y^*) \leq 0 \quad \forall k;$$

$$x_k^* [\nabla_{x_k} \theta_k(x^*, y^*)] = 0 \text{ and } y_k^* [\nabla_{y_k} \theta_k(x^*, y^*)] = 0 \quad \forall k,$$

where $(x_k^*, y_k^*) \in \tilde{T}$.

Note that theorem 2 develops a relationship between a Nash equilibrium solution and the KKT conditions. Having established the relationship, we use the results to estimate revenue or profit maximizing benchmark frontiers as described below.

3. Revenue Maximization Model

Consider a firm with fixed input levels wanting to maximize revenues by adjusting its output level. We describe a production process with a vector of inputs used to generate a single output and then generalize it to a multiple-output production process. We illustrate both cases with an example from the productivity literature.

3.1 Single-output model

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3 This is consistent with an output-oriented efficiency analysis in the productivity literature.
We estimate a production function with a single output and identify a Nash equilibrium solution using the MCP. Each firm adjusts its output level $y_r$ to maximize the revenue function $R_r$.\(^4\) Formulation (2) represents the revenue maximization model. To endogenously determine the price level, we define the inverse demand function $P(Y)$. In general, this demand function need only be strictly decreasing in $Y$. Since the market price in our model is affected by the total supply quantity $Y = \sum_k x_k + y_r$, we obtain the optimal output level as $y_r^* = \arg\max_r R_r$. The model is feasible while $P(Y) \geq 0$ and $y_r \geq 0$ (Farahat and Perakis, 2010) and can be estimated as follows:

$$
R^* = \max_{y_r} \left\{ \sum_r P(Y)y_r \left| \begin{array}{l}
\sum_k \lambda_{rk} y_k \geq y_r \quad \forall r; \\
\sum_k \lambda_{rk} x_{ki} \leq x_{ri} \quad \forall i, r; \\
\sum_k \lambda_{rk} = 1 \quad \forall r; \\
\lambda_{rk} \geq 0 \quad \forall k, r;
\end{array} \right. \right\}
$$

Defining $y_i$ as a random variable of quantity supplied in the market, we need a generalized form for the price function, $P(y_i)$, to estimate the inverse demand function. If the inverse demand function is strictly decreasing and continuously differentiable, then the revenue function is concave and continuously differentiable, and a Nash equilibrium solution exists (Murphy et al., 1982). For illustrative purposes, we assume a linear inverse demand function which satisfies these properties, i.e. $P(Y) = p_0 - \alpha Y$, where $p_0$ is a positive intercept and $\alpha$ indicates the nonnegative price sensitivity with respect to $Y$. (See appendix for a detailed discussion of the inverse demand function and the use of instrumental variables.) If $\alpha = 0$, then the price is constant regardless of the output level consistent with the standard analysis of allocative efficiency in the productivity and efficiency literature (Fried et al., 2008), i.e. the price is exogenous as in the case of perfect competition.

In the single-output revenue model (2) with a linear inverse demand function, we use the CP to find the Nash equilibrium solution. We define the Lagrangian function as:

$$L_r(y_r, \lambda_{rk}, \mu_1, \mu_2, \mu_3) = \sum_r P(Y)y_r - \sum_r \mu_1 (y_r - \sum_k \lambda_{rk} y_k) - \sum_r \mu_2 (\sum_k \lambda_{rk} x_{ki} - x_{ri} - r \mu_3 (k \lambda_{rk} - 1)).$$

The MCP is:

\(^4\) This is consistent with a profit maximization model, given fixed input prices and levels.
If the MCP gives the solutions $P(Y) < 0$, or $y_r < 0$, i.e. the inverse demand function returns a negative value, or the production output level is less than zero, this Nash equilibrium solution is inconsistent with production theory. Clearly, the sales price of a product cannot be negative. Similarly, if production will cause a profit loss, a firm’s best strategy is to shut down, i.e. the output level will be zero. Thus, we show that a Nash solution satisfies these two properties.

**Lemma 2**: A Nash solution to MCP problem (3) will satisfy $y_r \geq 0$ and $P(Y) \geq 0$.

Given $P^0 > 0$ and $\alpha \geq 0$, a small $\alpha$ means that a change in quantity of output will not affect the price significantly, but a large $\alpha$ will greatly affect the price. If the industry output level changes, the price will drop significantly and the revenues for all firms will likely decrease. Therefore, the firms have an incentive to restrict production to keep the price – and revenues – high. The same output level chosen by all firms is characterized by a common output level $\bar{y}_r$. The revenue maximizing benchmarks constitute a Nash equilibrium. Figure 3 illustrates the relationship between a Nash equilibrium and single-input-single-output production function, given parameter $\alpha$.

**Theorem 3**: If $P(Y) = P^0 - \alpha Y \geq 0$ and $\alpha$ is a small enough positive parameter, the Nash equilibrium solution is for all firms to produce on the production frontier.

**Theorem 4**: If $P(Y) = P^0 - \alpha Y \geq 0$ and $\alpha$ is a large enough positive parameter, the MCP will lead to a benchmark output level with $y_r = \bar{y}_r$ close to zero, where $\bar{y}_r$ defines a truncated output level.
We select a dataset from Dyson et al. (1990) describing a set of distribution centers for a large supermarket organization to illustrate the single-output NEP. The two inputs are stocks and wages. The outputs correspond to the activities of the distribution center (DC). The three output variables available are: 1) number of issues representing deliveries to supermarkets, 2) number of receipts in bulk from suppliers, and 3) number of requisitions to suppliers. In this illustrative example, we only use the number of issues as a single output variable and assume a simple inverse demand function \( P(Y) = 100 - \alpha Y \). Table 1 shows the best strategy for output expansion or contraction, given different price sensitivity values, \( \alpha \). As discussed, a firm’s best strategy is to produce on the production frontier if the \( \alpha \) value is small; alternatively, as \( \alpha \) increases the benchmark function becomes truncated. Note that regardless of the value of \( \alpha \), the price and output quantity are always larger than zero as stated in lemma 2.
Table 1 Nash equilibrium in single-output production

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</table>

3.2 Multiple-output Model

To build a demand function for multiple differentiated substitutable products, we use the affine demand function proposed by Farahat and Perakis (2010) and define it as

$$Y_q(p_q, p_{(-q)}) = Y_q^0 - \gamma_{q,q}p_q + \sum_{h \neq q} \gamma_{q,h}p_h$$

for all $q$, where $Y_q \geq 0$ and $p_{(-q)} \equiv (p_1, \ldots, p_{q-1}, p_{q+1}, \ldots, p_Q)$. For our purposes we define an inverse affine demand function, $Y_q^{-1}$, which exists if the condition of diagonal dominance of $\gamma$ matrix is satisfied (Bernstein and Federgruen, 2004). Specifically, we consider a linear inverse (indirect) affine demand function as

$$P_q(Y_q, Y_{(-q)}) = P_q^0 - \alpha_{q,q}Y_q - \sum_{h \neq q} \alpha_{q,h}Y_h$$

for all $q$, where $P_q \geq 0$, $Y_q = \sum_k y_{kq}$, $Y_{(-q)} \equiv (Y_1, \ldots, Y_{q-1}, Y_{q+1}, \ldots, Y_Q)$, and $\alpha_{q,q}$ is the diagonal element of output $q$ in the price sensitivity matrix $\alpha$. In particular, $P_q > 0$ is not a prerequisite constraint in the revenue maximization problem and can be relaxed. Below we define a set of properties and the conditions for relaxing
Four important properties of the price sensitivity matrix $\alpha$ are:\(^5\)

1) Weak diagonal dominance (WDD): if matrix $\alpha$ satisfies diagonal dominance then the revenue function is strictly concave as discussed above.

2) Moderate diagonal dominance (MDD): if matrix $\alpha$ satisfies $\alpha_{qq} \gg \sum_h \alpha_{qh}$ for all $q$. This property holds for product $q$ if the main effect $\alpha_{qq}$ caused by the same product is more intense than the minor effect $\alpha_{qh}$ created by another substitute product.

3) Symmetric matrix: a symmetric matrix $\alpha$ implies an equivalent bidirectional effect between any two substitute products.

4) Strong diagonal dominance (SDD): $\alpha_{qq} \gg \sum \alpha_{q} \sum \alpha_{h} \alpha_{qh}$ for all $q$, where $\sum \alpha_{q}$ denotes the sum of all elements in matrix $\alpha$ and $\alpha_{qh}$ denotes the trace which represents the sum of the elements on the diagonal of matrix $\alpha$. SDD means that each product’s quantity level generates a powerful main effect on the product’s price.\(^6\)

The WDD property is likely to be true because, in general, the price of product $A$ is more likely to be affected by the quantity produced of $A$ than by the quantity produced of the substitute product $B$. We use the following formulation (4) to identify the optimal output levels:

$$R^* = \max_{y_{rq}} \left\{ \sum_r \sum_q P_q(y_q, y_{(-q)}) y_{rq} \right\}$$

$$\left\{ \begin{array}{l}
\Sigma \lambda_{rk} y_{kq} \geq y_{rq} \forall q, r; \\
\Sigma \lambda_{rk} x_{ki} \leq x_{ri} \forall i, r; \\
\lambda_{rk} = 1 \forall r; \\
\lambda_{rk} \geq 0 \forall k, r;
\end{array} \right\} \quad (4)$$

Note that to identify a Nash equilibrium, the objective function has to be a strictly concave function in all arguments. Let $R = \sum_r \sum_q P_q(y_q, y_{(-q)}) y_{rq}$, giving

$$\frac{\partial R}{\partial y_{rq}} = P_q(y_q, y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_h \alpha_{hq} y_{rh}, \forall q, \quad \text{and} \quad \frac{\partial^2 R}{\partial y_{rq} \partial y_{rh}} = -\alpha_{qh} - \alpha_{qh}, \forall q, h.$$ 

A negative definite Hessian matrix will imply a strictly concave revenue function. Thus, the necessary and sufficient conditions are $\alpha_{qh} > 0$ and the price sensitivity matrix $\alpha$ satisfies the

---

\(^5\) Note that all output variables need to be normalized in data pre-processing to eliminate unit dependence.

\(^6\) For a discussion of the relationship among these properties see the weak, moderate, and strong dominance section in the appendix.
WDD property, namely, \( \alpha_{jj} > \sum_{h \neq j} \alpha_{jh} \) for all \( j \).

To solve the Nash equilibrium of formulation (4), we construct the complementary problem and define the Lagrangian function as:

\[
L_r\left(y_{rq}, \lambda_{rk}, \mu_1_{rq}, \mu_2_{ri}, \mu_3_r\right) = \sum_r \sum_q p_q(Y_q, Y_{(-q)})y_{rq} - \sum_r \sum_q \mu_1_{rq} \left(y_{rq} - \sum_k \lambda_{rk} Y_{kq}\right) - \sum_r \sum_i \mu_2_{ri} \left(\sum_k \lambda_{rk} X_{ki} - X_{ri}\right) - \sum_r \mu_3_r (\sum_k \lambda_{rk} - 1).
\]

The MCP is:

\[
0 = \frac{\partial L_r}{\partial y_{rq}} = \left(p_q(Y_q, Y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_1_{rq}\right) \perp y_{rq} \quad \forall r, q
\]

\[
0 \geq \frac{\partial L_r}{\partial \lambda_{rk}} = \left(\sum_q \mu_1_{rq} Y_{kq} - \sum_i \mu_2_{ri} X_{ki} - \mu_3_r\right) \perp \lambda_{rk} \geq 0 \quad \forall r, k
\]

\[
0 \geq \left(y_{rq} - \sum_k \lambda_{rk} Y_{kq}\right) \perp \mu_1_{rq} \geq 0 \quad \forall r, q
\]

\[
0 \geq \left(\sum_k \lambda_{rk} X_{ki} - X_{ri}\right) \perp \mu_2_{ri} \geq 0 \quad \forall r, i
\]

\[
0 = \left(\sum_k \lambda_{rk} - 1\right) \quad \forall r
\]

Similar results can now be developed for the multiple output case in theorem 5.

**Theorem 5**: If the price sensitivity matrix \( \alpha \) satisfies WDD but is not necessarily symmetric, then the MCP (6) generates \((X_{ri}, y_{rq}) \in \bar{T}\) where \(y_{rq}\) will approach the efficient frontier for small enough values of \(\alpha_{qq}; y_{rq} = \bar{y}_{rq}\) is the truncated benchmark output level that approaches zero as \(\alpha_{qq}\) approaches infinity.

**Corollary 1**: If the price sensitivity matrix \( \alpha \) satisfies the MDD property and \( \alpha_{qq} \gg \alpha_{hh}, q \neq h \), then the solution to the MCP (6) will satisfy \( y_{rq} < y_{rh} \ \forall r, q \).

\(^7\) If matrix \( \alpha \) does not satisfy the SDD property, the resulting Nash equilibrium solution may include \(y_{rq} < 0\). In this case formulation (5) is changed in the first inequality to state \(0 \geq \frac{\partial L_r}{\partial y_{rq}}\), and \(y_{rq} \geq 0\):

\[
0 \geq \frac{\partial L_r}{\partial y_{rq}} = \left(p_q(Y_q, Y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_1_{rq}\right) \perp y_{rq} \geq 0 \quad \forall r, q
\]

\[
0 \geq \frac{\partial L_r}{\partial \lambda_{rk}} = \left(\sum_q \mu_1_{rq} Y_{kq} - \sum_i \mu_2_{ri} X_{ki} - \mu_3_r\right) \perp \lambda_{rk} \geq 0 \quad \forall r, k
\]

\[
0 \geq \left(y_{rq} - \sum_k \lambda_{rk} Y_{kq}\right) \perp \mu_1_{rq} \geq 0 \quad \forall r, q
\]

\[
0 \geq \left(\sum_k \lambda_{rk} X_{ki} - X_{ri}\right) \perp \mu_2_{ri} \geq 0 \quad \forall r, i
\]

\[
0 = \left(\sum_k \lambda_{rk} - 1\right) \quad \forall r
\]
Theorem 5 is important because the relationship between price sensitivity matrix \( \mathbf{a} \) and the Nash equilibrium solution that can be identified from the characteristic of matrix \( \mathbf{a} \) gives insights into the Nash equilibrium regarding the elements in matrix \( \mathbf{a} \). The more price sensitive the product the more likely a firm will hold back production in order to increase its revenue.

Even if a large \( \alpha_{qq} \) results in a truncated benchmark production level, it does not necessarily result in a common output value for all firms, because some firms may be limited by the production frontier. Referring to figure 3, \( X_r \) is the smallest input value to generate the truncated benchmark output level. Note that the production processes using an input quantity between 0 and \( X_r \) will identify a benchmark on the production frontier. Without loss of generality and \( \alpha_{\infty} > 0 \) from MCP (6), we have:

\[
0 < y_{rq} \leq \frac{p_q^0 - \alpha_{qq} \sum_{k=1}^n y_{kq} - \sum_{h \neq q} \alpha_{hq} Y_h - \sum_{h \neq q} \alpha_{hh} Y_{rh}}{2\alpha_{qq}} 
\]

If for product \( q \) of firm \( r \) the efficient output level \( y_{rq} \) is lower than the truncated level \( \bar{y}_q \), that is, the production frontier limits output level \( y_{rq} \), then \( y_{rh} \) can exceed the truncated benchmark level \( \bar{y}_h \) for some product \( h \), because \( y_{rq} \) is smaller than the truncation level \( \bar{Y}_q \), and \( \frac{\alpha_{hq}}{\alpha_{hh}} \) and \( \frac{\alpha_{qh}}{\alpha_{hh}} \) do not go to zero in the inequality show in equation (7). \(^8\) Simply stated, firms will adjust their mix in output space to maximize revenues and generally some variation from the truncated benchmark production level may exist. \(^9\)

Again, we use our two-output illustrative example from the dataset described in section 3.1. The two output variables are the number of issues and the number of receipts, and the two inputs are stocks and wages. The inverse demand functions for issues and receipts are:

\[
P_{q_1} (Y_{q_1}, Y_{(-q_1)}) = 100 - \alpha_{q_1 q_1} Y_{q_1} - \alpha_{q_1 q_2} Y_{q_2}
\]

and

\[
P_{q_2} (Y_{q_2}, Y_{(-q_2)}) = 50 - \alpha_{q_2 q_1} Y_{q_1} - \alpha_{q_2 q_2} Y_{q_2}
\]

respectively. Table 2 reports the Nash equilibrium solution to the MCP (6) for different price sensitivity matrix \( \mathbf{a} \), all of which satisfy the WDD property. Once more a firm’s best strategy is to produce as close to the efficient frontier as possible for products with an insensitive inverse demand function implied by smaller values in the diagonal components of the \( \mathbf{a} \) matrix shown in

\(^8\) Note the exchange of \( q \) and \( h \).

\(^9\) This result is illustrated in table 2, case 2, DC 5.
case 1. As $\alpha_{qq}$ becomes larger the benchmark output level is truncated and approaches zero with respect to product $q$. In cases 2 the parameter $\alpha_{q_1q_1}$ is larger than case 1, the output $q_1$ decreases and output $q_2$ increases to maximize revenue. Similar in case 3, $\alpha_{q_2q_2}$ is increased relative to case 1 and the output $q_2$ decreases. In case 4 the parameter $\alpha_{q_2q_2}$ increases with respect to case 2, the solution shows output $q_2$ decreases to the truncated benchmark level. Increasing $\alpha_{q_1q_1}$ in cases 5 and 6, output $q_1$ approaches zero even though the $\alpha$ matrices do not satisfy the symmetric condition. In cases 7 and 8 $\alpha_{q_1q_1} = 2\alpha_{q_2q_2}$ and the results indicate that the ratio of output levels $q_1$ and $q_2$ are influence not only by the ratio of $\alpha_{q_1q_1}$ to $\alpha_{q_2q_2}$, but also by their absolute levels.

Table 2 Nash equilibrium in two-output production

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<th>2</th>
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Note that case 6 in Table 2, the price of product $q_1$ (issues) is less than 0, an unreasonable negative price, yet the revenue function is still equal to zero because $Y_{q_1} = 0$. Adding another constraint to restrict the price to be larger than zero will cause the quantity of product $q_2$ to drop, which results in a worse outcome.\(^{10}\)

### 4. Generalized Profit Maximization Model

This section defines a short-run profit model of an oligopolistic output market with a limited capacity input market, we only change the variable inputs, e.g., capital stock for production is fixed and employment or materials vary with demand (Marshall, 1920). Stigler (1939) argues that the quantitative variations of output can be described via the law of diminishing returns and marginal productivity theory when holding constant all but one of the productive factors and adjusting the quantity of the remaining factor. Thus, our generalized model treats fixed inputs and variable inputs separately.

This section also looks at the case of variable input markets with limited capacity and oligopolistic output markets, assuming that the inverse supply function of inputs and the inverse demand function of output are linear (see section 3 and the appendix). We formulate our generalized profit maximization model as equation (8):

$$
P^* = \max_{y_{rq}, x_{rj}} \left\{ \sum_r \sum_q p_q^r (Y_q, Y_{(-q)}) y_{rq} - \sum_r \sum_j p_j^{x^r} (x_j^r, x_{(-j)}^r) x_{rj}^r \right\} \left\{ \begin{array}{l}
\sum_k \lambda_{rk} Y_{kq} \geq y_{rq} \quad \forall q, r; \\
\sum_k \lambda_{rk} X_{kl}^F \leq X_{rl}^F \quad \forall i, r; \\
\sum_k \lambda_{rk} X_{kj}^V \leq x_{rj}^V \quad \forall j, r; \\
\sum_k \lambda_{rk} = 1 \quad \forall r; \\
\lambda_{rk} \geq 0 \quad \forall k, r; 
\end{array} \right\} \quad (8)
$$

where $X_{rl}^F$ is the data for fixed input $i$ and $x_{rj}^V$ is the decision variable for variable input $j$ of firm $r$. $Y_q = \sum_k q Y_{kq} + y_{rq}$ \(\forall q\) and $p_q^r (Y_q, Y_{(-q)}) = p_q^{r0} - \alpha_{qq} Y_q - \sum_h \alpha_{qh} Y_h \quad \forall q$ indicate the overall quantity and price of the inverse demand function of output product $q$ in the market. Similarly, for variable input $j$ the overall quantity $X_j^V = \sum_k x_{kj}^V + x_{rj}^V \quad \forall j$ and the inverse supply function $p_j^{x^r} (x_j^r, x_{(-j)}^r) = p_j^{x0} + \beta_{jj} x_j^V + \sum_{l \neq j} \beta_{jl} x_{rl}^V \quad \forall j$. Note that the objective function ignores the fixed input cost $\sum_l p_l^{x^F} (x_l^F, x_{(-l)}^F) x_{rl}^F$ since it is a constant sunk cost.

\(^{10}\) The intuition for case 6 can be built using the single-output case considering only product $q_2$. The related problem of negative demand in demand function is modified using the price mappings (described in Shubik and Levitan (1980), Soon, et al. (2009), and Farahat and Perakis (2010)).
To verify the existence and uniqueness of a solution, the profit function should be strictly concave. Let the profit function be \(PF = \sum_r \sum_q p_q^r (y_q, y_{(-q)}) y_{rq} - \sum_r \sum_j p_j^x (x_j^r, x_{(-j)}) x_{rj}\). That is, the revenue function \(\sum_r \sum_q p_q^r (y_q, y_{(-q)}) y_{rq}\) should be strictly concave and the variable cost function \(\sum_r \sum_j p_j^x (x_j^r, x_{(-j)}) x_{rj}\) strictly convex. We have \(\frac{\partial PF}{\partial y_{rq}} = p_q^r (y_q, y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_h \pi_h y_{rh}, \forall q\), and \(\frac{\partial^2 PF}{\partial y_{rq} \partial y_{rh}} = -\alpha_{qh} - \alpha_{hq}, \forall q, h\). A negative definite Hessian matrix will imply a strictly concave revenue function. Thus, the necessary and sufficient conditions are \(\alpha_{qp} > 0\) and the price sensitivity matrix \(\alpha\) satisfies the WDD property, namely, \(\alpha_{qq} \geq \sum \pi_h y_{rh\forall q}\). Also, we have \(\frac{\partial^2 PF}{\partial x_{rj} \partial x_{rj}} = -\beta_{jl} - \beta_{lj}, \forall j, l\). A negative definite Hessian matrix will imply a strictly concave negative cost function. Similarly, the necessary and sufficient conditions are \(\beta_{jl} > 0\) and the price sensitivity matrix \(\beta\) satisfies the WDD property.\(^{11}\)

To solve for a Nash equilibrium associated with equation (8), the CP is built and the Lagrangian function defined as:

\[L_r (y_{rq}, x_{rj}^v, \lambda_{rk}, \mu_{1r}, \mu_{2r}, \mu_{3r}, \mu_{4r}) = \sum_r \sum_q p_q^r (y_q, y_{(-q)}) y_{rq} - \sum_r \sum_j p_j^x (x_j^r, x_{(-j)}) x_{rj} - \sum_r \sum_q \mu_{1r} (y_{rq} - \sum_k \lambda_{rk} y_{kq}) - \sum_r \sum_j \mu_{2r} (\sum_k \lambda_{rk} x_{kli} - x_{rj}) - \sum_r \sum_j \mu_{3r} (\sum_k \lambda_{rk} x_{kji} - x_{rj}) - \sum_r \mu_{4r}(\sum_k \lambda_{rk} - 1).\]

The MCP is:

\[\begin{align*}
0 &\geq \frac{\partial L_r}{\partial y_{rq}} = (p_q^r (y_q, y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_h \pi_h y_{rh} - \mu_{1r}) \perp y_{rq} \geq 0 \forall r, q \\
0 &\geq \frac{\partial L_r}{\partial x_{rj}} = (-p_j^x (x_j^r, x_{(-j)}) - \beta_{jl} x_{rj} - \sum_l \beta_{lj} x_{rl} + \mu_{3r}) \perp x_{rj} \geq 0 \forall r, j \\
0 &\geq \frac{\partial L_r}{\partial \lambda_{rk}} = (\sum_k \mu_{1r} y_{kq} - \sum_k \mu_{2r} x_{kli} - \sum_k \mu_{3r} x_{kji} - \mu_{4r}) \perp \lambda_{rk} \geq 0 \forall r, k \\
0 &\geq (y_{rq} - \sum_k \lambda_{rk} y_{kq}) \perp \mu_{1r} \geq 0 \forall r, q \\
0 &\geq (\sum_k \lambda_{rk} x_{kli} - x_{rj}) \perp \mu_{2r} \geq 0 \forall r, i \\
0 &\geq (\sum_k \lambda_{rk} x_{kji} - x_{rj}) \perp \mu_{3r} \geq 0 \forall r, j \\
0 &\geq (\sum_k \lambda_{rk} - 1) \perp \forall r
\end{align*}\]

\(^{11}\) In a special case in which input markets are perfectly competitive \(\beta_{jl} = 0\), the inverse supply function will be constant and the cost function becomes a linear function. This does not affect the optimality condition, i.e. the profit function is still a strictly concave function if the revenue function is strictly concave.
The Nash equilibrium solution generated from MCP (9) exists and is unique when the price sensitivity matrices \( \alpha \) and \( \beta \) satisfy the WDD property. See section 4.1 for a similar proof.

In a perfectly competitive market, the profit efficient firms, i.e. achieving maximum profits (Farrell, 1957), must be allocatively efficient by using the least cost mix of inputs to produce the maximum revenue mix of outputs, and technically efficient by generating the most outputs with their level of inputs, (Färe, et al., 1994). In oligopolistic markets, however, profit maximization can be achieved without technical efficiency, i.e. rational inefficiency. We will continue to refer to the profit maximizing production possibility as allocatively efficient because it is not possible to change either the input mix or the output mix to increase profits. MCP (9) generates an allocatively efficient Nash solution.

**Theorem 6**: Given arbitrary price sensitivity matrices \( \alpha \) and \( \beta \) that satisfy WDD, MCP (9) generates all allocatively efficient Nash solutions \( (x^F_{ri}, x^F_{rj}, y^r_{pq}) \in \tilde{T} \). These solutions are on the frontier including the weakly efficient frontier\(^{12} \), but excluding the portion of the frontier associated with positive slacks and dual variables equal to zero on the input constraints.

Theorem 6 implies that the Nash equilibrium benchmark generated from MCP (9) exists on the production frontier using the same or fewer inputs than at least one anchor point, (Bougnol and Dulá, 2009).\(^{13} \) Based on theorem 5, if \( \alpha_{qq} \) becomes large, the production level will approach zero with respect to \( q \) and the Nash solution will be located on the weakly efficient portion of the production frontier which uses minimal input levels. In other words, if the price sensitivity to output is large enough, the Nash equilibrium benchmark suggests that a firm should operate on the weakly efficient portion of the frontier where more output can be generated using the same level of inputs. In this case, note that the profits are maximized by operating inefficiently, motivating the connection to rational inefficiency.

---

\(^{12} \) Weakly efficient frontier is defined as the portion of the input (output) isoquant along which one of the inputs (outputs) can be reduced (expanded) while holding all other netputs constant and remaining on the isoquant; see Färe and Lovell (1978) for more details.

\(^{13} \) Bougnol and Dulá (2009) propose a procedure to identify *anchor points* and show that if a point is an anchor point, then increasing an input or decreasing an output generates a new point on the free-disposability portion of the production possibility set.
The illustrative example of the generalized profit model also uses the dataset in section 3. The two output variables are the number of issues and the number of receipts, and the two variable inputs are stocks and wages. One fixed input is randomly generated from a Uniform [3,10] distribution. The inverse demand functions for issues and receipts are

\[ P_{q_1}(Y_{q_1}, Y_{(-q_1)}) = 100 - \alpha q_1 Y_{q_1} - \alpha Q_{q_2} Y_{q_2} \]

and

\[ P_{q_2}(Y_{q_2}, Y_{(-q_2)}) = 50 - \alpha Q_{q_2} Y_{q_2} - \alpha Q_{q_1} Y_{q_1} \]

respectively. The inverse supply functions for stocks and wages are

\[ P_{x_1}(X_{x_1}, X_{(-x_1)}) = 50 + \beta_{x_11} X_{x_1} + \beta_{x_12} X_{x_2} \]

and

\[ P_{x_2}(X_{x_2}, X_{(-x_2)}) = 30 + \beta_{x_21} X_{x_1} + \beta_{x_22} X_{x_2} \]

respectively. Table 3 reports the Nash equilibrium solution to MCP (9) for the price sensitivity matrices \( \alpha \) and \( \beta \), all of which satisfy the WDD property. Again, for outputs with insensitive inverse demand functions implied by smaller values in the diagonal components of the \( \alpha \) matrix, a firm’s best strategy is to produce near the efficient frontier; as \( \alpha_{qq} \) becomes large, the production approaches zero with respect to \( q \) as shown in case 4. Similarly on the input side, for inputs with sensitive inverse supply functions implied by larger values in the diagonal components of the \( \beta \) matrix, the best strategy is to use smaller input levels to produce on the weakly efficient frontier; as \( \beta_{jj} \) becomes smaller, the input level of the Nash equilibrium solution grows larger. However, the price sensitivity value \( \beta \) will affect the price of Nash solution significantly, cost will increase quickly and profits will drop. Cases 1, 2 and 3 show that as \( \beta_{jj} \) increases, costs also increase and producers have less incentive to produce. Cases 4 and 5 decrease the output level due to changes in the \( \alpha \) matrix; in particular, case 5 illustrates rational inefficiency because firms hold back producing additional output in order to maximize profits.
### Table 3: Nash equilibrium in two-output two-variable-input production

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<tr>
<th>Case</th>
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### 4.1 Existence and Uniqueness

If a Nash equilibrium does not exist, there is no purpose in talking about its properties, identification, etc. Further, if multiple equilibria exist, it is not clear which might result in any particular case. In this section we prove the existence and uniqueness of the Nash equilibrium solution identified by the MCP.

**Theorem 7:** MCP (9) generates a Nash equilibrium solution $(\mathbf{x}^F, \mathbf{x}^V^*, \mathbf{y}^*) \in \overline{T}.$
To get a unique Nash equilibrium, a strictly concave profit function is assumed. Given a convex production possibility set, theorem 8 states the uniqueness of the Nash equilibrium.

**Theorem 8**: If the profit function is a strictly concave function on \((X^F, x^V, y) \in \bar{T}\) that is continuous and differentiable and the price sensitivity matrices \(\alpha\) and \(\beta\) satisfy the WDD property, then the Nash equilibrium solution found using MCP (9) is unique if a solution exists for the maximization problem.

### 4.2 Price Sensitivity and Returns to Scale

It is necessary to understand the relationship between the price sensitivity matrices \(\alpha\) and \(\beta\) and the returns to scale (RTS) properties of the Nash equilibrium benchmarks. To address RTS properties, we must first identify the most productive scale size (MPSS). The production frontier is characterized by three regions: constant returns to scale (CRS), increasing returns to scale (IRS), and decreasing returns to scale (DRS). The MPSS can be identified for firm \(r\)’s input and output mix using the input-oriented CRS DEA technique formulated in (10). If the sum of \(\sum_k \lambda^{CRS}_{rk} = 1\) in the input-oriented CRS DEA\(^{14}\), we can identify such observations as operating at the MPSS (Banker, 1984):

\[
\min_{\theta_r, \lambda^{CRS}_{rk}} \left\{ \sum_r \theta_r \left| \begin{array}{c}
\sum_k \lambda^{CRS}_{rk} Y_{kq} \geq Y_{rq} \forall q, r; \\
\sum_k \lambda^{CRS}_{rk} X^F_{kl} \leq X^F_{rl} \forall i, r; \\
\sum_k \lambda^{CRS}_{rk} \lambda^{CRS}_{kj} \leq \theta_r x^V_{rj} \forall j, r; \\
\lambda^{CRS}_{rk} \geq 0 \forall k, r;
\end{array} \right. \right\} \quad (10)
\]

Let \(k^{MPSS}\) denote the set of observations having the MPSS property, one for each firm \(r\) in the dataset, and let \(y^*_{rq}\) and \(x^V_{rj}\) be the Nash equilibrium solutions obtained from MCP (9). Using these additional observations as the reference set, optimization problem (11) can be used to identify the returns to scale property for each production plan in the Nash solution.

\[
\min_{\theta_r, \lambda^{CRS}_{rk}} \left\{ \sum_r \theta_r \left| \begin{array}{c}
\sum_k \lambda^{CRS}_{rk}^{MPSS} Y_{k^{MPSS}q}^{MPSS} \geq y^*_{rq} \forall q, r; \\
\sum_k \lambda^{CRS}_{rk}^{MPSS} X^F_{k^{MPSS}l} \leq X^F_{rl} \forall i, r; \\
\sum_k \lambda^{CRS}_{rk}^{MPSS} x^V_{k^{MPSS}j} \leq \theta_r x^V_{rj} \forall j, r; \\
\lambda^{CRS}_{rk}^{MPSS} \geq 0 \forall k, r;
\end{array} \right. \right\} . \quad (11)
\]

\(^{14}\) Note there are potential multiple optimal solutions. See Zhu (2000) for additional details.
For each Nash solution of firm $r$, if $\sum k^{MPSS} \lambda^{CRS^+}_{r,k^{MPSS}} < 1$, firm $r$ operates under increasing returns to scale; if $\sum k^{MPSS} \lambda^{CRS^+}_{r,k^{MPSS}} > 1$, firm $r$ operates under decreasing returns to scale; or if $\sum k^{MPSS} \lambda^{CRS^+}_{r,k^{MPSS}} = 1$, firm $r$ operates under constant returns to scale. The equation $\sum k^{MPSS} \lambda^{CRS^+}_{r,k^{MPSS}}$ is termed the RTS index (RTSI).

For a one-input one-output production process, figure 4 depicts the true production function as a solid curve, the CRS estimated frontier as a straight dashed line, the VRS estimated frontier as a piece-wise linear bold dashed line, and the MPSS as Point B. In particular, based on theorem 6 the Nash equilibrium generated from MCP (9) should be located on the bold dashed lines. $X_A$ and $X_E$ are the upper and lower bounds for the variable input level.

**Corollary 2**: Assume all input and output variables are normalized to eliminate unit dependence, and the price of outputs dominates the price of inputs to ensure a positive marginal profit. Given a production frontier including three portions: IRS, CRS, and DRS, the MCP (9) generates a

In (9) note that both inputs and outputs are defined as adjustable; thus, all Nash equilibria are located on the production frontier. If this is not the case, for example if there are adjustment costs when changing input levels (Choi et al., 2006), then some equilibria may not be on the frontier as shown in figure 3. For these equilibria RTS are not defined because RTS is a frontier property. In (11) if $\theta_r$ is not equal to 1, then RTS is not defined for that production possibility; see for example Seiford and Zhu (1999) or Ray (2010).
Nash equilibrium solution that is characterized by DRS when the inverse demand and supply functions are less sensitive, or the Nash equilibrium is characterized by IRS when the inverse demand and supply functions are more sensitive.

Extending the illustrative example in section 2, we use formulation (11) to identify the RTS property of the Nash equilibrium solution shown in table 3; DCs 3, 10, 12, 15, and 19 are identified as operating at MPSS. Table 4 shows the RTS associated with the Nash solutions for cases 1 through 5 in table 3. Based on corollary 2, the sensitivity of output and sensitivity of input are the two oppositional forces in terms of scale. Case 1 represents a baseline and the Nash solutions present CRS or DRS properties. The sensitivity parameter of the supply function in case 2 increases relative to case 1, which encourages firms to hold back on the consumption of inputs, i.e. more DCs operate at MPSS in case 2. If we further increase the sensitivity parameter of the supply function, all DCs operate at MPSS in case 3. Case 4 results in all firms operating at MPSS or IRS, by increasing the sensitivity parameter of the demand function and leaving the sensitivity parameter of the supply function parameter the same as in case 1. Case 5 shows that all firms operate at IRS and on the weakly efficient portion of the frontier. This demonstrates the concept of rational inefficiency.
4.3 Allocative Efficiency and Directional Distance Function

The Nash equilibrium identified by using (9) is an allocatively efficient solution as shown in theorem 6. Zofio and Prieto (2006) suggest choosing the direction in the direction distance function (DDF) to move towards the allocatively efficient point. We extend their suggestion to the case of oligopolistic markets and suggest that each firm should select the direction for improvement in the DDF to move towards its Nash equilibrium benchmark.

The DDF as defined by Chambers et al. (1996; 1998) is the simultaneous contraction of inputs and expansion of outputs:

\[
\overline{D}_T(X^F, X^V, Y; g^{x^V}, g^v) = \max \{\delta \in \mathcal{R}: (X^F, X^V + \delta g^{x^V}, Y + \delta g^v) \in T\}
\]  

Table 4 Returns to scale of Nash equilibrium

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<tr>
<th>Case</th>
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<th>(q_2)</th>
<th>(j_1)</th>
<th>(j_2)</th>
<th>(q_1)</th>
<th>(q_2)</th>
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Returns to Scale

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* C, D and I indicate constant, decreasing, and increasing returns to scale respectively.
where $\delta$ is the distance measure and $\mathbf{g}^{x^r}, \mathbf{g}^y$ are the direction vectors for variable inputs and outputs respectively. Recall that since we do not change the fixed inputs in the short run, no direction is associated with them. We estimate the DDF for firm $r$ as:

$$\overline{D}_T(\mathbf{X}_r, \mathbf{X}_r^F, \mathbf{Y}_r, \mathbf{g}^{x^r}, \mathbf{g}^y) = \max_{\delta_r, \lambda_k} \left\{ \begin{array}{l}
\sum_k \lambda_k Y_{kq} \geq Y_{rq} + \delta_r g^y_q \quad \forall q; \\
\sum_k \lambda_k X_{kl}^F \leq X_{ri} \quad \forall i; \\
\sum_k \lambda_k X_{kj}^V \leq X_{rj}^V + \delta_r g^x_j \quad \forall j; \\
\sum_k \lambda_k = 1; \\
\lambda_k \geq 0 \quad \forall k;
\end{array} \right\}. \quad (13)$$

Because the method for selecting a direction $(\mathbf{g}^{x^r}, \mathbf{g}^y)$ is an open issue, the direction $(-\mathbf{1}, \mathbf{1})$ is usually chosen for simplicity. Alternatively, Frei and Harker (1999) determine the least-norm projection from an inefficient firm to the frontier, but this direction is non-proportional and is not unit-invariant. Färe et al. (2011) estimate an endogenous direction, but it is void of economic meaning. Therefore, we propose that firms’ direction for improvement move towards the allocatively efficient benchmark identified by the Nash equilibrium. Thus, the direction is firm-specific and can be calculated by following the equation for firm $r$:

$$\left(\mathbf{g}^{x^r}_r, \mathbf{g}^y_r\right) = \frac{(\mathbf{x}^{V^*}_r - \mathbf{x}^V_r, \mathbf{y}^*_r - \mathbf{y}_r)}{\left\| (\mathbf{x}^{V^*}_r - \mathbf{x}^V_r, \mathbf{y}^*_r - \mathbf{y}_r) \right\|} \quad (14)$$

where $\mathbf{x}^{V^*}_r$ and $\mathbf{y}^*_r$ are the benchmarks determined by the Nash equilibrium, $\mathbf{X}^V_r$ and $\mathbf{Y}_r$ are the vectors of the current variable input and output production, and $\left\| \cdot \right\|$ is the Euclidean norm. This ratio imposes $(\mathbf{g}^{x^r}, \mathbf{g}^y)$ is a unit vector.\(^{16}\)

Extending the example in table 3, case 1, we calculate the direction of improvement associated with this example as shown in table 5. The results indicate that when trying to maximize overall economic efficiency\(^{17}\) using formulation (8) it is not necessary to contract the variable inputs and expand the outputs. To maintain higher price and profit maximization, firm $r$ may achieve economic efficiency by changing its mix to become allocatively efficient. However, no firm takes a direction which increases all variable inputs and decreases all output levels as this would lead to a loss in profit.

\(^{16}\) The length of the directional vector influences the efficiency estimates in the DDF; the use of a unit vector has also been used in Fare et al. (2011).

\(^{17}\) Economic efficiency is the product of allocative efficiency and technical efficiency, see for example Fried et al. (2008).
5. Conclusion
This paper analyzes endogenous prices in productivity analysis. Given inverse demand and supply functions, a Nash equilibrium solution corresponding to profit maximization production plan within the production possibility set is identified using a mixed complementary problem (MCP). When the inverse demand and supply functions are constant functions, the standard analysis of efficiency assuming perfect competition and exogenous prices follows. For markets in which demand is heavily influenced by the total supply quantity, firms seek to decrease their output levels and maintain higher product prices to maximize profits. The proposed MCP model integrates oligopolistic market equilibrium and productivity analysis. We find that the resulting Nash equilibrium is an example of rational inefficiency.

Deviating from standard economic analysis, we consider the production limitations estimated from observed data and interpret the Nash equilibrium as the benchmark, or the
production plans each of the firms should work towards for more profitable production. Our work extends the efficiency literature on demand functions by considering multiple output production and allowing both outputs and variable inputs to be adjusted by the firm. Prior work primarily focused on individual firms decisions without consideration for the other firms in the market.

The identification of a unique Nash equilibrium allows further insights to operational improvement strategies. We show the relationship between price sensitivity and returns to scale in the Nash equilibrium. Based on the concept of allocative efficiency, we conclude that the Nash equilibrium is a useful guide for determining direction in the directional distance function.

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References


Appendix

Section 1: Supporting Discussions

Instrumental Variables

The inverse demand function in an oligopolistic market depends on the output levels of all firms. However, in some cases other factors influence the price. Three cases are discussed below. In the body of the paper we assume the simplest case, (1) estimating the inverse demand function with no omitted variables, 

\[ P_l = P^0 - \alpha Y_l + \varepsilon_l, \]

and \( E(Y_l \varepsilon_l) = 0, \varepsilon_l \sim N(0, \sigma^2) \) i.i.d. and a regression model using Ordinary Least Squares (OLS) and 

\[ E(P_l | Y_l = Y_l) = P^0 - \alpha Y_l. \]

However, if there exist omitted variables \( W_l \) which affect price \( P_l \), such that 

\[ P_l = P^0 - \alpha Y_l + \beta W_l + \eta_l \]

where \( \eta_l \sim N(0, \sigma^2) \), alternative cases need to be considered. In case (2), OLS can still provide consistent estimates when 

\[ E(Y_l \eta_l) = 0, \ E(W_l \eta_l) = 0 \]

and the quantity variable \( Y_l \) is uncorrelated with the omitted variable, i.e. \( E(Y_l W_l) = 0 \). Thus, the regression generates 

\[ E(P_l | Y_l = Y_l) = P^0_W - \alpha Y_l, \]

where \( P^0_W = P^0 + \beta E(W_l) \). In case (3), if \( Y_l \) is correlated with the omitted variable \( E(Y_l W_l) \neq 0 \), let 

\[ \varepsilon_l = \beta W_l + \eta_l, \]

which results in \( E(Y_l \varepsilon_l) \neq 0 \).

Case 3 is termed endogeneity in econometrics; OLS provides inconsistent, biased, and inefficient estimates for the \( \alpha \) parameters of interest (Greene, 2011). To address this issue, we use an instrumental variable \( Z_l \) that is highly correlated with \( Y_l \) but independent of \( W_l \) and \( \eta_l \), specifically 

\[ E(Z_l W_l) = 0, \ E(Z_l \eta_l) = 0. \]

The regression model can be rewritten as 

\[ P_l = P^0 - \alpha Z_l + \beta W_l + \eta_l, \]

and OLS can provide consistent estimates and 

\[ E(P_l | Z_l = z_l) = P^0_W - \alpha z_l. \]

As described in section 3.1, our paper focuses on an inverse demand function expressed and estimated by a linear function 

\[ P(Y) = P^0 - \alpha Y, \]

where \( P^0 \) is the intercept corresponding to case 1. However, if endogeneity exists in the inverse demand model, we can identify instrumental variables using the methods described in Goldberger (1972), Morgan (1990), and Angrist and Krueger (2001). Note that when output quantity changes, we assume a change in supply curve rather than a change in quantity supplied.

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18 The omitted variable could be the price or quantity of substitute products or other contextual factors that could affect the price of output \( q \).
Weak, Moderate, and Strong Dominance Properties

**Lemma 3:** In the two-output product case, if matrix $\alpha$ satisfies MDD and symmetric properties, then matrix $\alpha$ satisfies SDD.

Proof: If matrix $\alpha$ satisfies MDD and symmetric properties, it will spontaneously lead to the transitivity property, which implies that the main effect of each product dominates the minor effect of the other products, i.e. the SDD property.

**Lemma 4:** If price sensitivity matrix $\alpha$ satisfies the SDD property, then solving MCP (5) generates a solution such that $y_{rq} \geq 0$ and $P_q(Y_q, Y_{(-q)}) \geq 0$ where $\forall (X_{rl}, y_{rq}) \in \bar{T}$.

Proof: $P_q(Y_q, Y_{(-q)}) = \alpha_{qq}y_{rq} - \sum_{h \neq q} \alpha_{hq}y_{rh} - \mu_{1q} = 0$, that is $P_q(Y_q, Y_{(-q)}) \geq \alpha_{qq}y_{rq} + \sum_{h \neq q} \alpha_{hq}y_{rh}$. If $y_{rq}, y_{rh} \geq 0, q \neq h$, then $P_q(Y_q, Y_{(-q)}) \geq 0$ and the revenue function is nonnegative. The case $y_{rq} \geq 0, y_{rh} < 0, q \neq h$ will not happen because, given $P_q(Y_q, Y_{(-q)}) = P_0^q - \alpha_{qq}Y_q - \sum_{h \neq q} \alpha_{qh}Y_h$ the increase of revenue through the increase of price $P_q(Y_q, Y_{(-q)}) \geq 0$ cannot exceed the decrease of revenue through the increase of price $P_h(Y_h, Y_{(-h)}) \geq 0$ when $y_{rh}$ becomes smaller and smaller and $y_{rh} < 0$ since $P_q(Y_q, Y_{(-q)})$ is not sensitive with respect to $y_{rh}$ by symmetric $\alpha$ and $\alpha_{qq} \gg \alpha_{qh}$ for all $q$. Thus, the preferred solution for revenue maximization problem is $y_{rh} = 0$. Similar to the impossible case $y_{rq} < 0, y_{rh} \geq 0, q \neq h$, by the SDD property we have $P_q(Y_q, Y_{(-q)}) \geq 0$ which will cause $y_{rq} = 0$ to maximize revenue. If $y_{rq} < 0, y_{rh} < 0, q \neq h$, then we have $P_q(Y_q, Y_{(-q)}) > 0$. Similar to lemma 2, we have solution $y_{rq} = y_{rh} = 0$ to maximize the revenue function $\sum_q P_q(Y_q, Y_{(-q)})y_{rq} > 0$. Therefore, the case $y_{rq} < 0, y_{rh} < 0$ will not happen. Moreover, if price sensitivity matrix $\alpha$ is symmetric and satisfies the WDD property, and $\alpha_{qq} \gg \alpha_{qh}$ for all $q$, then, as lemma 3, solving MCP (5) will automatically generate $y_{rq} \geq 0$ and $P_q(Y_q, Y_{(-q)}) \geq 0$ where $\forall (X_{rl}, y_{rq}) \in \bar{T}$. 

Lemma 4 shows a case of SDD of sensitivity matrix $\alpha$. The WDD or MDD properties are not enough to ensure $y_{rq} \geq 0$ in MCP (5). That is, if $\alpha$ is not symmetric or violates MDD, then for some $y_{rq}$ a Nash equilibrium solution may set $y_{rq} < 0$. We illustrate this in two cases below.

Case 1: If price sensitivity matrix $\alpha$ satisfies MDD but not symmetry and we know that if $y_{rq} \neq 0$, then $P_q(Y_q, Y_{(-q)}) - \alpha_{qq} y_{rq} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq} = 0$. We have $y_{rq} = \frac{P_q(Y_q, Y_{(-q)}) - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{\alpha_{qq}} = \frac{P_q^0 - \alpha_{qq} y_{rq} - \sum_{h \neq q} \alpha_{qh} y_{rh} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{\alpha_{qq}}$. Finally, $y_{rq} = \frac{P_q^0}{2\alpha_{qq}} - \frac{\sum_{k \neq r} \alpha_{kh} y_{kh} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{2\alpha_{qq}}$. Thus, $y_{rq}$ might be less than zero and $P_q(Y_q, Y_{(-q)}) < 0$ for the revenue maximization problem as $\alpha_{hq} \gg \alpha_{qq}$ and $y_{rh} > 0$.

Case 2: If price sensitivity matrix $\alpha$ satisfies WDD and symmetric properties, in an extreme case, if for one product $q$, we have $\frac{\alpha_{qh}}{\alpha_{qq}} \to 1^-$, this notation means the ratio approaches 1 from the left-hand side. We know $y_{rq} = \frac{P_q(Y_q, Y_{(-q)}) - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{\alpha_{qq}} = \frac{P_q^0 - \alpha_{qq} y_{rq} - \sum_{h \neq q} \alpha_{qh} y_{rh} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{\alpha_{qq}}$. Then, $y_{rq} \approx \frac{P_q^0}{2\alpha_{qq}} - \frac{\sum_{k \neq r} \alpha_{kh} y_{kh} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{2\alpha_{qq}}$. Thus, $y_{rq}$ might be less than zero and $P_q(Y_q, Y_{(-q)}) > 0$ for the revenue maximization problem since $\alpha_{qq}$ and $\alpha_{hh}$ are large, and $\alpha_{qq} > \alpha_{hh}$ and $y_{rh} > 0$.

Therefore, to ensure $y_{rq} \geq 0$ from formulation (5), the SDD property provides a sufficient condition based on lemma 4. If matrix $\alpha$ satisfies SDD and given an extreme case, for all $q$, $\frac{\sum(\alpha) - \text{tr}(\alpha)}{\alpha_{qq}} \to 0^+$ this notation means the ratio approaches 0 from the right-hand side. If we know $y_{rq} = \frac{P_q(Y_q, Y_{(-q)}) - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{\alpha_{qq}}$, then we can obtain an estimate of $y_{rq} = \frac{P_q^0 - \alpha_{qq} \sum_{k \neq r} \alpha_{kh} y_{kh} - \sum_{h \neq q} \alpha_{hq} y_{rh} - \mu_{1rq}}{2\alpha_{qq}} \approx \frac{P_q^0 - \alpha_{qq} \sum_{k \neq r} \alpha_{kh} y_{kh} - \mu_{1rq}}{2\alpha_{qq}} > 0$ for the revenue maximization problem.
Cost minimization case

In the case of a single fixed input and a single variable input and a given output level, figure 5 illustrates the Nash equilibrium solution obtained by minimizing costs. Each firm attempts to adjust its variable input to reach the isoquant, holding a fixed input constant in the short run.

![Figure 5 Adjusted variable input in Nash equilibrium](image)

We construct a multi-input cost model to identify a Nash equilibrium solution using MCP. The result shows that the Nash equilibrium solution is on the production frontier regardless of the matrix selected. In particular, to formulate the MCP with multiple variable inputs, first we define the Lagrangian function as:

\[
L_r(x^V_{ij}, \lambda_{rk}, \mu_{1r}, \mu_{2r}, \mu_{3r}, \mu_{4r}) = -\sum_r \sum_j P^V_j (x^V_j, x^V_{-j}) x^V_{ij} - \sum_r \sum_q \mu_{1rq} (y_{rq} - \sum_k \lambda_{rk} y_{kq}) - \sum_r \sum_i \mu_{2ri} (\sum_k \lambda_{rk} x^F_{ki} - x^F_{ri}) - \sum_r \sum_j \mu_{3rj} (\sum_k \lambda_{rk} x^V_{kj} - x^V_{rj}) - \sum_r \mu_{4r} (\sum_k \lambda_{rk} - 1).\]

Then the resulting MCP problem is:

\[
0 \geq \frac{\partial L_r}{\partial x^V_{ij}} = (-P^V_j (x^V_j, x^V_{-j}) - \beta_{ij} x^V_{ij} - \sum_l \beta_{lj} x^V_{ri} + \mu_{3r}) x^V_{ij} \geq 0 \quad \forall r, j
\]

\[
0 \geq \frac{\partial L_r}{\partial \lambda_{rk}} = (\sum_k \mu_{1rk} y_{kq} - \sum_i \mu_{2ri} x^F_{ki} - \sum_j \mu_{3rj} x^V_{kj} - \mu_{4r}) \lambda_{rk} \geq 0 \quad \forall r, k
\]

\[
0 \geq (y_{rq} - \sum_k \lambda_{rk} y_{kq}) \mu_{1rq} \geq 0 \quad \forall r, q
\]

\[
0 \geq (\sum_k \lambda_{rk} x^F_{ki} - x^F_{ri}) \mu_{2ri} \geq 0 \quad \forall r, i
\]

\[
0 \geq (\sum_k \lambda_{rk} x^V_{kj} - x^V_{rj}) \mu_{3rj} \geq 0 \quad \forall r, j
\]

\[
0 = (\sum_k \lambda_{rk} - 1) \quad \forall r
\]
**Theorem 9:** In the cost minimization case a Nash equilibrium generated from MCP (15) exists on the production frontier, given an arbitrary $\beta$ matrix with all nonnegative components satisfying WDD.

Proof: Proving the existence of a Nash equilibrium is similar to theorem 7. The Nash equilibrium generated from MCP will stay on the production frontier, given an arbitrary $\beta$ matrix satisfying WDD. If an equilibrium output vector exists and $x_{rj}^V > 0$, then it must satisfy the first order condition of MCP. From the complementary condition, we have the following first order condition:

$$-P_j^{x^V}(x^V_r, x^V_{(-j)}) - \beta_{ij} x_{rj}^V - \sum_{l \neq j} \beta_{ij} x_{rl}^V + \mu 3_{rj} = 0 \quad \forall r,j$$

which can be expressed in matrix notation as:

$$-P^{x^V} - \beta x^V e - \beta^T x^V + \mu 3_r = 0 \quad \forall r$$

where $x^V$ is a matrix with $(x^V_1, ..., x^V_K)$ and each vector $x^V_k = (x^V_{k1}, ..., x^V_{kJ})^T$. $e$ is a vector $(1, ..., 1)^T$ with $K$ elements. $P^{x^V}$ is a price vector with elements $P_j^{x^V}$. $\mu 3_r$ is a vector of the Lagrangian multiplier with elements $\mu 3_{rj}$. If $x^{V*}$ is the solution obtained from the first order condition, we need to show that $P_j^{x^V}(x^{V*}, x^{V*}_{(-j)}) = P_j^{x^V} + \beta_{ij} x_{rj}^{V*} + \sum_{l \neq j} \beta_{ij} x_{rl}^{V*} \geq 0$ for all $j$.

We express this equation in the matrix notation $P^{x^V} - \beta x^{V*} e$. Obviously, the first order condition gives $P^{x^V} + \beta x^{V*} e + \beta^T x^{V*} - \mu 3_r > 0$ if $P^{x^V} > 0$ and $\beta$ have nonnegative elements. This implies that it is necessary to set $(\sum_k \lambda_{rk} x^{V*}_{kj} - x_{rj}^{V*}) = 0$ in terms of MCP; the upper bound of input level is characterized by the least value at the free disposability hull of inputs and the lower bound is the input level described by the free disposability hull of outputs shown in theorem 6. Because $(\sum_k \lambda_{rk} x^{V*}_{kj} - x_{rj}^{V*}) = 0$, that is, for cost minimization, the whole quantity of supply market would be minimized to reach a lower price at the inverse supply function, a firm’s best strategy is to reduce its input level and to produce on the production frontier.
Generalized profit model as revenue maximization case

In a special case of the revenue model, we assume that the output level directly follows the variable input, namely, the level of variable input determines and controls the level of output. For example, in the semiconductor manufacturing industry raw silicon wafers are released into the production line to generate the actual die output. If the yield is 100%, the output level is a linear function of the variable input level. Assuming a constant unit cost of the variable input, we formulate the profit maximization model as:

\[
P_F^* = \max_{y_{rq}} \left\{ \sum_r \sum_q P_q^r(Y_q, Y_{(-q)})y_{rq} - \sum_r \sum_q f_q^r(Y_q, Y_{(-q)})y_{rq} \right\}
\]

where \( P_q^r(Y_q, Y_{(-q)}) \) becomes a constant and presents a unit cost of variable input \( x_{rj}^Y = \rho y_{rq} \), and \( \rho \) is a coefficient to change the units to a linear function. Intuitively, model (16) is quite similar to formulation (4), the revenue maximization model. The profit function \( \sum_r \sum_q [P_q^r(Y_q, Y_{(-q)}) - P_q^r(Y_q, Y_{(-q)})]y_{rq} \) is a concave function because \( P_q^r(Y_q, Y_{(-q)}) \) is a constant and \( P_q^r(Y_q, Y_{(-q)})y_{rq} \) is a linear function. Thus, a Nash equilibrium exists and is unique. See sections 3 and 4 in the body of the paper.

Section 2: Proofs

**Lemma 1:** Let output levels be decision variables denoted by \( y_{rq} \) as output \( q \) of firm \( r \) and \( y_{rq} \geq 0 \); further, let input levels be decision variables denoted by \( x_{ri} \) as input \( i \) of firm \( r \), \( x_{ri} \geq 0 \), and \( (x_{ri}, y_{rq}) \in T \). Define \( P_q^r(y_{rq}) \) as a concave function of \( y_{rq} \) and assume that either the inverse demand function \( P_q^r(y_{rq}) \) is a non-increasing or a convex function of \( y_{rq} \). Thus, for each \( Y_{(-r)q} > 0 \), where \( Y_{(-r)q} = \sum_{k \neq r} x_{kq} \), \( P_q^r(y_{rq} + Y_{(-r)q})y_{rq} \) is a concave function of \( y_{rq} \) for \( y_{rq} \geq 0 \). Similarly, let \( P_i^X(x_{ri} + X_{(-r)i})x_{ri} \) be a convex function of \( x_{ri} \) for \( x_{ri} \geq 0 \), where \( X_{(-r)i} = \sum_{k \neq r} x_{ki} \) and \( P_i^X(x_{ri}) \) is an inverse supply function. Further, if either \( P_q^r(y_{rq}) \) is strictly decreasing or is strictly convex, then \( P_q^r(y_{rq} + Y_{(-r)q})y_{rq} \) is a strictly concave function on the nonnegative \( y_{rq} \geq 0 \) and \( \sum_q P_q^r(y_{rq} + Y_{(-r)q})y_{rq} - \sum_i P_i^X(x_{ri} + X_{(-r)i})x_{ri} \) is a concave function on \( (x_{ri}, y_{rq}) \in T \).
Proof: Murphy et al. (1982) prove the single output product case that when \( y_r \geq 0 \) and \( Y_{(-r)} > 0 \), the revenue function \( R_r = P^r(y_r + Y_{(-r)})y_r \) is a concave function of \( y_r \) for \( y_r \geq 0 \) on the nonnegative real line since \( \frac{\partial^2 R_r}{\partial y_r^2} < 0 \). In our special case of Murphy et al. proven in their lemma 1, the production possibility set \((x, y)\) is a convex set and the boundary is a piece-wise linear concave function which characterizes a production function with diminishing returns. Thus, \( P_q^r(y_{rq} + Y_{(-r)q})y_{rq} \) is a concave function of \( y_{rq} \) for \( y_{rq} \geq 0 \) and \((x_{ri}, y_{rq}) \in \bar{T} \) since, given fixed input levels, firm \( r \) can expand output only by increasing \( y_{rq} \). Similarly, we can prove a convex cost function of \( i^{th} \) input resource \( P_i^X(x_{ri} + X_{(-ri)})x_{ri} \) and concave profit function \( \sum_q P_q^r(y_{rq} + Y_{(-r)q})y_{rq} - \sum_i P_i^X(x_{ri} + X_{(-ri)})x_{ri} \).

**Theorem 1:** If the profit function of firm \( r \), \( \theta_r(x_{ri}, y_{rq}) = \sum_q P_q^r(Y_q)y_{rq} - \sum_i P_i^X(X_i)x_{ri} \) is concave with respect to \((x_{ri}, y_{rq})\) and continuously differentiable, where \( Y_q = \sum_k y_{kq} \) and \( X_i = \sum_k x_{ki} \), then \((x^*, y^*) \in \bar{T} \) is a Nash-Cournot oligopolistic market equilibrium if and only if it satisfies the set of VI \((F((x^*, y^*)), (x, y) - (x^*, y^*)) \geq 0, \forall (x, y) \in \bar{T} \). That is, \( \sum_k F_k((x^*, y^*))((x_k, y_k) - (x^*_k, y^*_k)) \geq 0 \) \( \forall (x_k, y_k) \in \bar{T} \), where

\[
F_k((x, y)) = (-\nabla_{x_k} \theta_k(x, y), -\nabla_{y_k} \theta_k(x, y)), \quad \nabla_{x_k} \theta_k(x, y) = \left( \frac{\partial \theta_k(x, y)}{\partial x_{k1}}, \ldots, \frac{\partial \theta_k(x, y)}{\partial x_{kd}} \right) \\
\nabla_{y_k} \theta_k(x, y) = \left( \frac{\partial \theta_k(x, y)}{\partial y_{k1}}, \ldots, \frac{\partial \theta_k(x, y)}{\partial y_{kq}} \right).
\]

Proof: To simplify the proof, we first focus on revenue function with a single output. If the revenue function \( P(Y) y_r \) is concave with respect to \( y_r \) and continuously differentiable, then \((x^*_k, y^*_d) \in \bar{T} \) is a Nash-Cournot oligopolistic market equilibrium if and only if it satisfies the set of VI \((F(y^*), y - y^*) \geq 0, \forall (x_{ki}, y_k) \in \bar{T} \). That is, \( \sum_k F_k(y^*)(y_k - y^*_k) \geq 0 \) \( \forall (x_{ki}, y_k) \in \bar{T}; \quad F_k(y^*) = -P(Y^*) - y^*_k P'(Y^*) \). Since the revenue function \( P(Y) y_r \) is a continuously differentiable function and concave with respect to \( y_r \), for a fixed \( r \), the Nash equilibrium condition \( P(y^*_r, y^*_r) y^*_r - P(y_r, y^*_r) y_r \geq 0 \), \( \forall (X_{ri}, y_r) \in \bar{T} \) is equivalent to the variational inequality problem \( F_r(y^*) (y_r - y^*_r) \geq 0 \), that is, \( \langle F_r(y^*), y_r - y^*_r \rangle \geq 0, \forall (X_{ri}, y_r) \in \bar{T} \). Then, summing over all firms \( k \) generates \( \langle F(y^*), y - y^* \rangle \geq 0 \), \( \forall (X_{ki}, y_k) \in \bar{T} \geq 0 \). This result can be
Theorem 2: Consider an oligopoly with $K$ firms, with an inverse demand function $P^Y(\cdot)$ that is strictly decreasing and continuously differentiable in $y$, and an inverse supply function $P^X(\cdot)$ that is strictly increasing and continuously differentiable in $x$. Since lemma 1 shows that the profit function $\theta_k(x_k, y_k)$ is concave and $\theta_k(\cdot, y) \geq 0$ for all $y$, then $(\mathbf{x}^*, \mathbf{y}^*) = ((\mathbf{x}^*_1, \mathbf{y}^*_1), (\mathbf{x}^*_2, \mathbf{y}^*_2), \ldots, (\mathbf{x}^*_K, \mathbf{y}^*_K))$ is a Nash equilibrium solution if and only if

$$\nabla_{x_k} \theta_k(\mathbf{x}^*, \mathbf{y}^*) \leq 0 \text{ and } \nabla_{y_k} \theta_k(\mathbf{x}^*, \mathbf{y}^*) \leq 0 \quad \forall k;$$

$$x_k^* \left[ \nabla_{x_k} \theta_k(\mathbf{x}^*, \mathbf{y}^*) \right] = 0 \text{ and } y_k^* \left[ \nabla_{y_k} \theta_k(\mathbf{x}^*, \mathbf{y}^*) \right] = 0 \quad \forall k$$

where $(\mathbf{x}^*_k, \mathbf{y}^*_k) \in \mathcal{T}$.

Proof: We derive the formulas above based on the KKT conditions. Note that the KKT conditions are both necessary and sufficient conditions for a unique global optimum since the model maximizes a strictly concave profit function over a convex polyhedral set (the production possibility set). The detail of existence and uniqueness of a Nash equilibrium is addressed in section 4 of the paper.

Lemma 2: A Nash solution to MCP problem (3) will satisfy $y_r \geq 0$ and $P(Y) \geq 0$.

Proof: $P(Y) - \alpha y_r - \mu_1 - r = 0$, that is $P(Y) \geq \alpha y_r$ since $\mu_1 \geq 0$. If $y_r \geq 0$, then $P(Y) \geq 0$ and the revenue function is nonnegative. If $y_r \leq 0$ and $P(Y) \geq 0$, a firm’s best strategy to maximize the revenue function is to make $y_r = 0$. The case $y_r \leq 0$ and $P(Y) < 0$ will not happen because if $P(Y) < 0$, then there exists at least one firm generating $y_k > 0, k \neq r$ such that $P(Y) = P^0 - \alpha Y < 0$. However, to maximize its revenue, firm $k$ prefers to produce $y_k = 0$. In other words, $P(Y) = P^0 - \alpha Y \geq 0$ if $y_r \leq 0$. In addition, if $\alpha$ is a large positive number, $y_r$ can be very small but positive to ensure a positive revenue function. Thus, any solution to this MCP (3) model enforces that $y_r$ and $P(Y)$ are nonnegative.

Theorem 3: If $P(Y) = P^0 - \alpha Y \geq 0$ and $\alpha$ is a small enough positive parameter, the Nash equilibrium solution is for all firms to produce on the production frontier.
Proof: In MCP, \( \frac{\partial L_r}{\partial y_r} = (P(Y) - \alpha y_r - \mu_1 Y) = 0, \forall r \); where \( \alpha \) is small enough, then \( P(Y) - \alpha y_r = \mu_1 Y \geq 0 \). In the extreme case, \( \alpha = 0 \), then \( P(Y) = P^0 = \mu_1 Y > 0 \). By MCP, \( 0 \geq (y_r - \sum_k \lambda_{rk} Y_k) \perp \mu_1 Y > 0, \forall r \), which gives \( y_r - \sum_k \lambda_{rk} Y_k = 0 \). Once again, a firm’s best strategy is to produce on the production frontier.

**Theorem 4:** If \( P(Y) = P^0 - \alpha Y \geq 0 \) and \( \alpha \) is a large enough positive parameter, the MCP will lead to a benchmark output level with \( y_r = \bar{y}_r \) close to zero, where \( \bar{y}_r \) defines a truncated output level.

Proof: Since \( P^0 - \alpha Y \geq 0 \) from lemma 2 and \( P^0 \) is a constant, then \( \leq \frac{P^0}{\alpha} \), meaning that a larger \( \alpha \) will result in a smaller \( Y \). In the MCP, \( 0 \geq (y_r - \sum_k \lambda_{rk} Y_k) \perp \mu_1 Y > 0, \forall r \). If \( Y \) is small, then \( (y_r - \sum_k \lambda_{rk} Y_k) < 0 \), i.e. \( \mu_1 Y = 0 \). In other words, we can increase \( \alpha \) until no firm would choose to produce on the production frontier in a Nash equilibrium solution, and then all \( \mu_1 Y = 0, \forall r \).

Proving this results in a truncated benchmark output level and requires us to show that if \( \alpha \) increases, then \( y_r \) decreases and approaches zero. Since \( \mu_1 Y = 0 \) and we know \( y_r \geq 0 \) by lemma 2, \( P(Y) - \alpha y_r = 0 \) in the MCP and \( y_r = \frac{P(Y)}{\alpha} \).

In addition, \( y_r = \frac{P(Y)}{\alpha} = \frac{P^0 - \alpha \sum_k \lambda_{rk} Y_k}{2\alpha} \). If there are only two firms in the market, \( y_1 = \frac{P^0 / \alpha - y_2}{2} \) and \( y_2 = \frac{P^0 / \alpha - y_1}{2} \), then \( y_1 = y_2 = \frac{P^0}{3\alpha} \). This constant identifies the truncation output level for production. If there are \( K \) firms in the market, \( y_r = \frac{P^0 - \alpha \sum_k \lambda_{rk} Y_k}{2\alpha} \) and \( y_r = \frac{(P^0 / \alpha) - (K-1)(P^0 / 2\alpha) + (K-1)2y_r + (K-1)2\alpha}{2} \), then \( \sum_k \lambda_{rk} Y_k = \frac{2}{K-2} \left( 2y_k - \frac{P^0}{\alpha} + (K-1) \left[ \frac{P^0}{2\alpha} - \left( \frac{K-1}{2} \right) y_r \right] \right) = \frac{2}{K-2} \left( \left( \frac{P^0}{2} \alpha \right) y_r + \left( \frac{P^0}{2\alpha} \right) (K-3) \right) \). We replace \( \sum_k \lambda_{rk} Y_k \) in equation \( y_r \), thus \( y_r = \frac{P^0 - \alpha \sum_k \lambda_{rk} Y_k}{2\alpha} = \frac{P^0}{(K+1)\alpha} - \frac{P^0}{(K+1)\alpha} = \frac{P^0}{(K+1)\alpha} \). Therefore, for \( K \) firms \( y_r = \frac{P^0}{(K+1)\alpha} = \bar{y}_r \) and this constant \( \bar{y}_r \) identifies the benchmark output level. As \( \alpha \) goes to infinity, \( y_r = \frac{P^0}{(K+1)\alpha} \to 0 \).

**Theorem 5:** If the price sensitivity matrix \( \alpha \) satisfies WDD but is not necessarily symmetric,
then the MCP (6) generates \((X_r, y_r) \in \bar{T}\) where \(y_r\) will approach the efficient frontier for small enough values of \(\alpha_{qq}\); \(y_r = \bar{y}_r\) is the truncated benchmark output level that approaches zero as \(\alpha_{qq}\) approaches infinity.

Proof: This is similar to theorems 3 and 4. We know \(\frac{\partial L_r}{\partial y_r} = P_q(Y_q, Y_{(-q)}) - \alpha_{qq}y_r - \sum_{h \neq q} \alpha_{hq}y_{rh} \leq \mu_1 \) if \(\alpha_{qq}\) value is small enough and we consider a special case \(\alpha_{qq} = 0\), and the \(\alpha\) matrix is diagonally dominant, then \(0 < P_q(Y_q, Y_{(-q)}) = P_q^0 \leq \mu_1 \). Referring to the MCP, \(0 \geq (y_{rq} - \sum_k \lambda_{rk} Y_{kq}) \perp \mu_1 \forall r, q\), meaning \(y_{rq} - \sum_k \lambda_{rk} Y_{kq} = 0\), or a firm’s best strategy is to produce on the production frontier except for the portion associated with positive slacks and dual variables equal to zero on the output constraints since increasing output does not affect the price reduction. On the other hand, if the \(\alpha_{qq}\) value is large enough, \(P_q(Y_q, Y_{(-q)}) = P_q^0 - \alpha_{qq} y_q - \sum_{h \neq q} \alpha_{hq}y_h \geq 0\) and \(P_q^0\) is a constant, then \(y_q \leq \frac{P_q^0 - \sum_{h \neq q} \alpha_{hq}y_h}{\alpha_{qq}}\). As \(\alpha_{qq}\) becomes larger, \(y_q\) approaches zero. Referring to the MCP, \(0 \geq (y_{rq} - \sum_k \lambda_{rk} Y_{kq}) \perp \mu_1 \forall r, q\). If \(Y_q\) is small, then \((y_{rq} - \sum_k \lambda_{rk} Y_{kq}) < 0\) and \(\mu_1 \) = 0. In other words, we can increase \(\alpha_{qq}\) until no firm would choose to produce on the production frontier in a Nash equilibrium solution, and then all \(\mu_1 \) = 0, \forall r, q. To show this result, as \(\alpha_{qq}\) increases, then \(y_{rq}\), the truncated output level becomes smaller and approaches zero. Since \(P_q(Y_q, Y_{(-q)}) - \alpha_{qq} y_q - \sum_{h \neq q} \alpha_{hq}y_{rh} - \mu_1 \leq 0\), and we know \(\mu_1 = 0\) and \(y_{rq} \geq 0\), thus \(P_q(Y_q, Y_{(-q)}) - \alpha_{qq} y_q - \sum_{h \neq q} \alpha_{hq}y_{rh} = 0\) in MCP (6) and \(0 \leq y_{rq} = \frac{P^0_q - \alpha_{qq} \sum_{k \neq r} y_{kq} - \sum_{h \neq q} \alpha_{hq} y_{hr} - \sum_{h \neq q} \alpha_{hq} y_{rh}}{2\alpha_{qq}}\). For the two-output products example, \(y_{r1} = \frac{p_{r1}^0}{2\alpha_{11}} - \frac{\alpha_{12}y_2}{2\alpha_{11}} y_{r2}\) and \(y_{r2} = \frac{p_{r2}^0}{2\alpha_{22}} - \frac{\alpha_{21}y_1}{2\alpha_{22}} - \frac{\alpha_{12}y_2}{2\alpha_{22}} y_{r1}\); replacing \(y_{r2}\)

in equation \(y_{r1}\) gives \(y_{r1} = \left(1 - \frac{\alpha_{12}y_2 + \alpha_{21}y_1}{4\alpha_{11}\alpha_{22}}\right) \left[\left(p_{r1}^0 - \frac{\alpha_{21}y_2}{4\alpha_{11}\alpha_{22}}\right) - \frac{1}{2} \left(\frac{\alpha_{21}}{4\alpha_{11}\alpha_{22}}\right)\sum_{k \neq r} y_{k2}\right]\). Also, \(y_{r2} = y_{r2} + \sum_{k \neq r} y_{k2}\) finally gives \(y_{r1} = \left(1 - \frac{\alpha_{12}y_2 + \alpha_{21}y_1}{4\alpha_{11}\alpha_{22}}\right) \left[\left(p_{r1}^0 - \frac{\alpha_{21}y_2}{4\alpha_{11}\alpha_{22}}\right) - \frac{1}{2} \left(\frac{\alpha_{21}}{4\alpha_{11}\alpha_{22}}\right)\sum_{k \neq r} y_{k1} + \left(\frac{\alpha_{12}}{4\alpha_{11}} + \frac{\alpha_{21}}{4\alpha_{11}}\right)\sum_{k \neq r} y_{k2}\right]\). Based on WDD, \(y_{r1} \approx \left(p_{r1}^0 - \frac{1}{2} \left(\frac{\alpha_{21}}{4\alpha_{11}}\right)\sum_{k \neq r} y_{k1} + \left(\frac{\alpha_{12}}{4\alpha_{11}}\right)\sum_{k \neq r} y_{k2}\right).\)
This result shows that $y_{r1}$ is a function of $y_{k1}$ and $y_{k2}$, not a variable of index $r$. Thus, $y_{r1}$ is limited by a truncated level $\bar{y}_{r1}$ for all firms, since for all firms $r$ the same equation applies as does $y_{r1}$ for revenue maximization. Similar equation can be derived for $y_{r2}$. In addition, as $\alpha_{11}$ approaches infinity $y_{r1} \approx \frac{-\sum_{k \neq r} y_{k1}}{2}$. That is, $y_{r1} = \bar{y}_{r1}$ should be equal to zero. We can extend this result to outputs of more than two. Therefore, the truncation point approaches zero as $\alpha_{qq}$ becomes large.

**Corollary 1**: If the price sensitivity matrix $\alpha$ satisfies the MDD property and $\alpha_{qq} \gg \alpha_{hh}, q \neq h$, then the solution to the MCP (6) will satisfy $y_{rq} < y_{rh} \forall r, q$.

Proof: Theorem 5 proves Corollary 1.

**Theorem 6**: Given arbitrary price sensitivity matrices $\alpha$ and $\beta$ that satisfy WDD, MCP (9) generates all allocatively efficient Nash solutions $(X^F_{ri}, x^V_{rij}, y^*_{r}) \in \tilde{T}$. These solutions are on the frontier including the weakly efficient frontier, but excluding the portion of the frontier associated with positive slacks and dual variables equal to zero on the input constraints.

Proof: Based on theorem 5, if $y_{rq} > 0$, then $x^V_{rij} > 0$ because there is no free lunch axiom in production theory (Färe et al., 1985). According to formulation (11) $-P^V_j (X^V_j, X^V_{(-j)}) - \beta_{jj} x^V_{rij} - \sum_{l \neq j} \beta_{lj} x^V_{rl} + \mu 3_{rj} = 0$, that is, $P^V_j (X^V_j, X^V_{(-j)}) + \beta_{jj} x^V_{rij} + \sum_{l \neq j} \beta_{lj} x^V_{rl} = \mu 3_{rj}$. Consider that $\beta_{jl} \geq 0$, $P^V_j > 0$ and the $\beta$ matrix is diagonally dominant; then $\mu 3_{rj} > 0$.

Referring to MCP (9), $0 \geq (\sum_{k} \lambda_{rk} X^V_{k} - x^V_{rij}) \perp \mu 3_{rj} > 0 \forall r, j$, which gives $\sum_{k} \lambda_{rk} X^V_{k} - x^V_{rij} = 0$. Based on theorem 5 we know that $\mu 1_{rq} \text{ might/might not be equal to zero for all } r, q$ according to the price sensitivity matrix $\alpha$, i.e. equation $y_{rq} - \sum_{k} \lambda_{rk} Y_{kq} \leq 0$. Thus, a firm’s best strategy is to adjust its variable input and output levels approaching the production frontier.

The solution becomes allocatively efficient. Further, $\sum_{k} \lambda_{rk} X^V_{k} - x^V_{rij} = 0$ implies that the slacks of the input constraints are equal to zero and the feasible region of the Nash solution is the production possibility set $\tilde{T}$ excluding the region for which the input level is larger than the least value at the free disposability hull of the inputs. This exception of the free disposability hull of
inputs implies an upper bound of adjustable input level. Note that the points on the free disposability hull of inputs, except the anchor points, have positive slacks and dual variables equal to zero on the inputs’ constraints. Therefore, all Nash equilibrium solutions \((X^F_{r_l}, x^V_{r_j}, y^*_{rq})\) belong to \(\tilde{T}\) excluding the input level larger than the anchor point at the free disposability hull of inputs.

**Theorem 7:** MCP (9) generates a Nash equilibrium solution \((X^F, x^V^*, y^*) \in \tilde{T}\).

Proof: If an equilibrium output vector exists and \(x_{rj} > 0, y_{rq} > 0\), it must satisfy the first order condition of MCP (9). The complementary condition gives the following first order condition on the output side:

\[
P_q(Y_q, Y_{(-q)}) - \alpha_{qq}y_{rq} - \sum_{h \neq q} \alpha_{hq}y_{rh} - \mu_{1, rq} = 0 \quad \forall r, q
\]

This condition can be expressed in matrix notation as:

\[
P^Y_0 - \alpha e - \alpha^T y_r - \mu_1 r = 0 \quad \forall r
\]

where \(y\) is a matrix with \((y_1, ..., y_K)\) and each vector \(y_k = (y_{k1}, ..., y_{kq})^T\). \(e\) is a vector \((1, ..., 1)^T\) with \(K\) elements. \(P^Y_0\) is a price vector with elements \(P^Y_q\). \(\mu_1 r\) is a vector of the Lagrangian multiplier with elements \(\mu_{1, rq}\). If \(y^*\) is the solution obtained from the first order condition, we need to show that \(P^Y_q(Y^*, Y_{(-q)}^*) = P^Y_0 - \alpha_{qq}y^*_q - \sum_{h \neq q} \alpha_{qh}y^*_h \geq 0\) for all \(q\). This equation can be expressed in matrix notation as \(P^0 - \alpha y^* e\). Obviously, the first order condition gives \(P^0 - \alpha y^* e = \alpha^T y_r^* + \mu_1 r \geq 0\) if \(y^*_r \geq 0\) for all \(r\) by lemma 2.

Similar to the first order condition on the variable input side

\[-P^X_0^V - \beta x^V e - \beta^T x^V_r + \mu_3 r = 0 \quad \forall r\]

where \(x^V\) is a matrix with \((x^V_1, ..., x^V_K)\) and each vector \(x^V_k = (x^V_{k1}, ..., x^V_{kj})^T\). \(P^X_0\) is a price vector with elements \(P^X_j\). \(\mu_3 r\) is a vector of the Lagrangian multiplier with elements \(\mu_{3, rj}\). If \(x^V^*\) is the solution obtained from the first order condition, we need to show that \(P^X_j(x^V^*, X_{(-j)}^*) = P^X_j + \beta_{jj}X_{jj}^* + \sum_{l \neq j} \beta_{jl}X_{lj}^* \geq 0\) for all \(j\). This equation can be expressed in matrix notation as \(P^X_j + \beta x^V e\). Obviously, the first order condition gives \(-\beta^T x^V_r + \mu_3 r = P^X_j + \beta x^V e \geq 0\) if \(x^V_r \geq 0\) for all \(r\) by the estimated production possibility set \(\tilde{T}\) describing a positive lower bound.
of input level. Therefore, if an equilibrium vector exists, it must equal \((x^*, y^*)\).

To show that \((x^*, y^*)\) is indeed an equilibrium vector, for any nonnegative vector \((X^F, \bar{x}^V, \bar{y}) \in \bar{T}\) where \((X^F, \bar{x}^V, \bar{y}) \neq (X^F, x^V, y^*)\), we consider \((X^F, \bar{x}^V, \bar{y})\) in which all the elements are equal to \((X^F, x^V, y^*)\) except for some \(x^r, y^r\) columns. We need to show that

\[
\sum_r \sum_q p_q^V(y_q, \bar{y}_{(-q)}) \bar{y}_{rq} - \sum_r \sum_j p_j^V(\bar{x}_j^V, \bar{x}_{(-j)}) \bar{x}_{rj}^V \leq \sum_r \sum_q p_q^V(y_q^*, Y_{(-q)}^*) y_{rq}^* - \sum_r \sum_j p_j^V(x_j^V, X_{(-j)}) x_{rj}^V
\]

for all \(r\). Since \(PF\) is a strictly concave function under concavity and differentiability assumptions for the maximization problem, and \((x^V, y^*)\) satisfies the first order condition and the KKT condition, then \((x^V, y^*)\) must be a global optimum, i.e. the complementary condition provides a Nash equilibrium solution: \(\sum_r \sum_q p_q^V(y_q, \bar{y}_{(-q)}) \bar{y}_{rq} - \sum_r \sum_j p_j^V(\bar{x}_j^V, \bar{x}_{(-j)}) \bar{x}_{rj}^V \leq \sum_r \sum_q p_q^V(y_q^*, Y_{(-q)}^*) y_{rq}^* - \sum_r \sum_j p_j^V(x_j^V, X_{(-j)}) x_{rj}^V\) for all \(r\) and \((X^F, \bar{x}^V, \bar{y}) \in \bar{T}\).

**Theorem 8:** If the profit function is a strictly concave function on \((X^F, x^V, y) \in \bar{T}\) that is continuous and differentiable and the price sensitivity matrices \(\alpha\) and \(\beta\) satisfy the WDD property, then the Nash equilibrium solution found using MCP (9) is unique if a solution exists for the maximization problem.

Proof: To prove the uniqueness, let two vectors \((X^F, \bar{x}^V, \bar{y})\) and \((X^F, x^V, y^*)\) be solutions and \((X^F, \bar{x}^V, \bar{y}) \neq (X^F, x^V, y^*)\) satisfy the variational inequality:

\[
\sum_k F_k(x^F, x^V, y^*)^T \cdot ((X^F_k, x^V_k, y^*_k) - (X^F_k, x^V_k, y^*_k)) \geq 0 \quad \forall (X^F_k, x^V_k, y^*_k) \in \bar{T}; \quad (17)
\]

\[
\sum_k F_k((X^F, \bar{x}^V, \bar{y}))^T \cdot ((X^F_k, \bar{x}^V_k, y^*_k) - (X^F_k, \bar{x}^V_k, \bar{y}_k)) \geq 0 \quad \forall (X^F_k, \bar{x}^V_k, y^*_k) \in \bar{T}; \quad (18)
\]

Substituting \(\bar{x}^V_k, \bar{y}_k\) for \(x^V_k, y^*_k\) in (17) and \(x^V_k^*, y^*_k\) for \(x^V_k^*, y^*_k\) in (18) and adding the resulting inequalities gives

\[
\sum_k (F_k((X^F, x^V, y^*)) - F_k((X^F, \bar{x}^V, \bar{y})))^T \cdot (X^F_k, \bar{x}^V_k, \bar{y}_k) - (X^F_k, x^V_k^*, y^*_k) \geq 0
\]

However, this inequality does not satisfy the definition of strict monotonicity. Thus, \(\bar{x}^V = x^V, \bar{y} = y^*\) and the solution is unique.
**Corollary 2:** Assume all input and output variables are normalized to eliminate unit dependence, and the price of outputs dominates the price of inputs to ensure a positive marginal profit. Given a production frontier including three portions: IRS, CRS, and DRS, the MCP (9) generates a Nash equilibrium solution that is characterized by DRS when the inverse demand and supply functions are less sensitive, or the Nash equilibrium is characterized by IRS when the inverse demand and supply functions are more sensitive.

Proof: Intuitively, for one-variable-input one-output production process if the inverse demand and supply functions are less sensitive, this is illustrated in a special case where both price sensitivity matrix $\alpha$ and $\beta$ are equal to zero; the profit function $PF$ can be written as maximizing $PF = Y_0y - P^V_0x^V$. Let $PF^*$ denote the optimal value of profit function. Thus, we can express the function $y = \frac{PF^* + P^V_0x^V}{pY_0}$, where $\frac{P^V_0}{pY_0}$ indicates the slope of profit function. Given the price of outputs dominating the price of inputs, in a special case the slope $\frac{P^V_0}{pY_0} \rightarrow 0^+$, the optimal solution of profit maximization problem will show the flat profit line tangent to the production possibility set. Since $\frac{P^V_0}{pY_0} \rightarrow 0^+$, a firm would like to generate a Nash solution on the DRS frontier for profit maximization based on theorems 5 and 6, i.e., in extreme case, the input level of the Nash solution has to be on the upper bound defined by the least value of the free disposability hull of inputs (see firm A in figure 4). DRS is associated with the insensitive inverse demand and supply function. Therefore, the Nash equilibrium solution generated from MCP (9) presents the DRS with respect to MPSS and $\sum_{k MPSS} \lambda^{CRS*}_{r_k MPSS} = \sum_k \lambda^{CRS*}_{r_k} > \sum_k \lambda^{*}_{r_k} = 1$. Similarly, we can show that the Nash solutions present the IRS when more sensitive inverse demand and supply functions occur, i.e., a profit function with larger slope. The result can be extended to the multiple-input multiple-output case.