ISEN 601
Location Logistics

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Rectilinear MF Location

- Total cost: \( f(\hat{x}_1, \ldots, \hat{x}_n) = f_1(\hat{x}_1, \ldots, \hat{x}_n) + f_2(y_1, \ldots, y_n) \)

- As in SF Location problem, can consider x and y coordinates separately!

- In x direction:
  \[ f_1(x_1, \ldots, x_n) = \sum_{l \in j \in k \in m} v_{jk} |x_j - x_k| + \sum_{j \in i \in l} u_{ji} |x_j - q_i| \]

- In y direction:
  \[ f_2(y_1, \ldots, y_n) = \sum_{l \in j \in k \in m} v_{jk} |y_j - y_k| + \sum_{j \in i \in l} u_{ji} |y_j - q_i| \]
Rectilinear MF Location

- Equivalent problem:

\[
\min \sum_{jk} v_{jk} (p_{jk} - q_{jk}) + \sum_{ij} u_{ij} (r_{ij} + s_{ij})
\]

s.t.
\[
\begin{align*}
    x_j - q_{jk} + f_{jk} &= x_k & j \leq k \leq n \\
x_j - r_{ij} - s_{ij} &= a_i & i \leq j \leq n \\
q_{jk}, f_{jk}, r_{ij}, s_{ij} &\geq 0
\end{align*}
\]

(in x coordinate)

Intersection Point Property

- There is an optimal solution such that the (individual) coordinates of the new facilities coincide with (individual) coordinates of existing facilities.
Multifacility Euclidean Distances

- Hyperboloid Approximation Procedure (HAP)
- Total cost function:
  \[ f(\bar{x}_1, \ldots, \bar{x}_n) = \sum_{i=1}^{m} w_i \ell_i(\cdot) + \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} \bar{\ell}_i(\cdot) \]
  analogous to the single-facility cost function, we can show that this cost function is convex in the coordinates.

Multifacility Euclidean Distances

- Define some notation:
  \[ \alpha_i = \sum_{j=1}^{n} \frac{v_{ij} x_j}{\bar{\ell}_2(\bar{x}_j, \bar{x}_i)} + \sum_{i=1}^{m} \frac{w_i a_i}{\bar{\ell}_2(\bar{x}_i, P_i)} \]
  \[ \beta_i = \sum_{j=1}^{n} \frac{v_{ij} y_j}{\bar{\ell}_2(\bar{x}_j, \bar{x}_i)} + \sum_{i=1}^{m} \frac{w_i b_i}{\bar{\ell}_2(\bar{x}_i, P_i)} \]
  \[ \Gamma_i = \sum_{j=1}^{n} \frac{v_{ij} x_j}{\bar{\ell}_2(\bar{x}_j, \bar{x}_i)} + \sum_{i=1}^{m} \frac{w_{ii}}{\bar{\ell}_2(\bar{x}_i, P_i)} \]

\[ \frac{\partial f}{\partial x_i} = \Gamma_i (\cdot) \frac{\partial}{\partial x_i} \]
\[ \frac{\partial f}{\partial y_i} = \Gamma_i (\cdot) y_i - \beta_i \]
\[ \frac{\partial f}{\partial P_i} = \Gamma_i (\cdot) P_i - \alpha_i \]
Multifacility Euclidean Distances

First-order condition:

\[
\chi - \frac{\partial \ell}{\partial \chi} = 0
\]

\[
\theta_l = \frac{\partial \ell}{\partial \theta_l}
\]

Multifacility Euclidean Distances

• HAP iterative procedure:

\[
x_{i+1} = \frac{\alpha \left( x_{1i}^k, ..., x_{ni}^k \right)}{\sum_{j=1}^{n} (x_{1j}^i, ..., x_{nj}^i)}
\]

\[Y_l \text{ are similar}\]

Algorithm:

Initial choice: \( x_1, y_1, ..., x_n, y_n \)

Iterate using \#

"HAP"
Multifacility Euclidean Distances

- Some observations:
  - Stopping criteria:
  - Convex hull property
  - All hold similar to the 1-facility Weber case

Recall: Multi-facility Location

- Locate 2 or more new facilities:
Location-Allocation Problems

Recall previous models:
Optimize location of new facilities, weights with existing facilities

Now:
Optimize location of new facilities and interaction with existing facilities

Application:
> Located Distribution Centers in a Sales Region
> Bank Branches
> Convenience Stores in a City

Location-Allocation Problems

Example:

What are the $w_i$? Transportation intensity between DCs & Retailers

$\equiv$ Number of truckloads per day
Location-Allocation Problems

A simple problem statement (location along a line) is given in:


(available on class web site)

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General Formulation

\[
\min \sum_{j=1}^{m} \sum_{i=1}^{n} w_{ji} \left( x_{ji} - \bar{x}_{ji} \right)^2 \]

\[
\text{Min } \sum_{j=1}^{m} \sum_{i=1}^{n} w_{ji} d_2 \left( x_{ji}, \bar{x}_{ji} \right)
\]

s.t. \[\sum_{j=1}^{m} w_{ji} = w_i \quad \forall i \rightarrow \text{must meet demand}\]

\[w_{ji} \geq 0\]

\[w_i \rightarrow \text{requirement in truck loads per day for retailer } i\]

\[x_{ji} \text{ and } w_{ji} \text{ are all decision variables}\]
General Formulation

- Assume there are no capacity constraints at the new facilities
- Any structural results?

![Diagram](image)

Rectilinear distances

- Method of full enumeration:
  - Worst case: \( m \) existing facilities
  - Then there are \( m^2 \) intersection points
  - If we need to place \( n \) new facilities then we have
  \[
  (m^2 \times m^2 \times \ldots \times m^2) = m^{2n} \text{ possibilities}
  \]
Euclidean distances

• Can't do full enumeration as in rectilinear case (why?)
• Employ heuristics
  – Not guaranteed to give optimal solution

• Heuristic idea:
  – Use location and allocation in an alternating fashion

Euclidean distances

• Alternating Location-Allocation (ALA) heuristic: Choose locations; allocate; calculate new locations; repeat.
  1. \( k=0 \); choose a starting set of locations \( x_j^0 \)
  2. Determine a number of sets \( S_j^k \) by assigning each existing facility to its closest new facility.
  3. Solve multiple single facility problems to get new \( x_s^k \) and \( S_j^k \)
  4. Assign new \( S_j^k \)s to the new \( x_s^k \)s

Until: \( |x_j^k - x_j^i| \leq \varepsilon \) for every small number \( \varepsilon \)
Euclidean distances

- Discussion of heuristic:
  - no guarantee of optimality
  - very sensitive to initial inputs
  - no guarantee to converge

⇒ run multiple scenarios w/ different starting points.

Location on Networks

- Consider a given transportation network
- New facilities located anywhere on the network
  - on edge or node
Definitions

**Path** → sequence of nodes \( v_1, v_2, ..., v_j \) in such a way that \( \{v_i, v_{i+1}\}, ..., \{v_{j-1}, v_j\} \) must be \( E \).

**Cycle** → path in which \( v_i = v_j \) for \( i \neq j \).

**Tree** → a network with no cycles.

Advantages of Network Models
Problems to Consider

- n-median:

- n-center:

Problems to Consider

- Covering:
Median Problems on Trees

- 2 general properties for tree networks:
- Property 1: Node optimality

Median Problems on Trees

- Proof:
Median Problems on Trees

- Property 2: Majority

Median Problems on Trees

- Proof:
Convexity on Trees

- Define distance between points x and y:

Convexity on Trees

- Convexity of $d(x,v)$:

- Convexity of $d(x,y)$: