Errata in Chapter 9

On page 555, part of Remark 23 the equation in the middle of the page:

\[ \tilde{H}_{ij}(0,w) = \sum_{k \in S, k \neq i} \tilde{H}_{kj}(0,w) \left( \frac{q_{ik}}{-q_{ii}} \right) \frac{q_{ik}}{q_{ik} + w} \]

should be changed to

\[ \tilde{H}_{ij}(0,w) = \sum_{k \in S, k \neq i} \tilde{H}_{kj}(0,w) \left( \frac{q_{ik}}{-q_{ii}} \right) \frac{-q_{ii}}{-q_{ii} + w}. \]

That directly affects the solution to Problem 92 on pages 563-566. However, it does not affect solution to problem 95 on page 572 where it is used.

Solution to problem 92

(changes marked in RED).

Solution. At \( t = 0 \) water level just crosses over from nominal to concerning. Using the same notation as in Problem ?? for \( X(t) \) and \( Z(t) \), we have \( X(0) = 20 \) and \( Z(0) = 1 \). Let \( T \) be the time when the water level crosses back to becoming nominal from concerning, i.e.

\[ T = \inf \{ t > 0 : X(t) = 20 \}. \]

To compute \( E[T] \), we follow the analysis in Remark ?? and use \( H_{ij}(x,t) = P \{ T \leq t, Z(T) = j \mid X(0) = x, Z(0) = i \} \) in particular its LST w.r.t. \( t \), \( \tilde{H}_{ij}(x,w) \). To obtain \( E[T] \), the expected number of days from \( t = 0 \) for the water level to return to nominal values, we use

\[ E[T] = (-1) \frac{d}{dw} \sum_{j=1}^{5} \tilde{H}_{1j}(20,w) \]

at \( w = 0 \). To compute \( \frac{d}{dw} \tilde{H}_{ij}(x,w) \) at \( w = 0 \) here too we consider a very small \( h > 0 \) and obtain it approximately as \( \frac{\tilde{H}_{ij}(x,h) - \tilde{H}_{ij}(x,0)}{h} \). Now, to evaluate \( \tilde{H}_{ij}(x,h) \) and \( \tilde{H}_{ij}(x,0) \), we can write down from Equation (??), for \( j = 1, 2, 3, 4, 5, \)

\[
\begin{bmatrix}
\tilde{H}_{1j}(x,w) \\
\tilde{H}_{2j}(x,w) \\
\tilde{H}_{3j}(x,w) \\
\tilde{H}_{4j}(x,w) \\
\tilde{H}_{5j}(x,w)
\end{bmatrix}
= a_{1,j}(w)e^{S_{1}(w)x}\phi_{1}(w) + a_{2,j}(w)e^{S_{2}(w)x}\phi_{2}(w) + \ldots + a_{5,j}(w)e^{S_{5}(w)x}\phi_{5}(w) \tag{1}
\]

where \( a_{i,j}(w), S_{j}(w) \) and \( \phi_{j}(w) \) values need to be determined for \( w = 0 \) and \( w = h \) for some small \( h \).

We can obtain \( S_{j}(w) \) for \( j = 1, 2, 3, 4, 5 \) as the scalar solutions to the characteristic equation

\[ det(DS(w) - wI + Q) = 0. \]

But this is identical to that in Problem ???. Likewise \( \phi_{j}(w) \) can be computed as the column vectors that satisfy

\[ S_{j}(w)D\phi_{j}(w) = (wI - Q)\phi_{j}(w) \]
which is also identical to that in Problem ?? . Thus refer to Problem ?? for \( \phi_j(w) \) and \( S_j(w) \) for 
\( j = 1, 2, 3, 4, 5 \) at \( w = 0 \) and \( w = h \). What remains in Equation (??) are the \( a_{i,j}(w) \) values for 
\( w = 0 \) and \( w = h \). For that refer back to the approach in Remark ?? . First of all

\[
\begin{bmatrix}
\tilde{H}_{1j}(x,w) \\
\tilde{H}_{2j}(x,w) \\
\tilde{H}_{3j}(x,w) \\
\tilde{H}_{4j}(x,w) \\
\tilde{H}_{5j}(x,w)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

for \( j = 1, 2 \) since the first passage time can never end in states 1 or 2 as the drift is negative in those 
states (only when the drift is positive is it possible to cross over into a particular buffer content 
level from below). Thus we only need \( a_{i,j}(w) \) values for \( i = 1, 2, 3, 4, 5 \) and \( j = 3, 4, 5 \).

Of those fifteen unknown \( a_{i,j}(w) \) values, nine can be obtained through the following boundary conditions:

\[
\begin{align*}
\tilde{H}_{33}(20, w) &= 1, \tilde{H}_{34}(20, w) = 0, \tilde{H}_{35}(20, w) = 0, \\
\tilde{H}_{44}(20, w) &= 1, \tilde{H}_{43}(20, w) = 0, \tilde{H}_{45}(20, w) = 0, \\
\tilde{H}_{55}(20, w) &= 1, \tilde{H}_{53}(20, w) = 0 \text{ and } \tilde{H}_{54}(20, w) = 0.
\end{align*}
\]

For the remaining six unknowns we use for \( j = 3, 4, 5 \)

\[
\begin{align*}
\tilde{H}_{1j}(0, w) &= \sum_{k=2}^{5} \tilde{H}_{kj}(0, w) \left( \frac{q_{1k}}{-q_{11}} \right) \frac{-q_{11}}{-q_{11} + w}, \\
\tilde{H}_{2j}(0, w) &= \tilde{H}_{1j}(0, w) \left( \frac{q_{21}}{-q_{22}} \right) \frac{-q_{22}}{-q_{22} + w} + \sum_{k=3}^{5} \tilde{H}_{kj}(0, w) \left( \frac{q_{2k}}{-q_{22}} \right) \frac{-q_{22}}{-q_{22} + w}
\end{align*}
\]

where \( q_{ij} \) corresponds to the element in the \( i^{th} \) row and \( j^{th} \) column of \( Q \).

Solving the above fifteen equations we get for \( w = 0 \),

\[
\begin{bmatrix}
a_{1,3}(0) & a_{2,3}(0) & a_{3,3}(0) & a_{4,3}(0) & a_{5,3}(0) \\
a_{1,4}(0) & a_{2,4}(0) & a_{3,4}(0) & a_{4,4}(0) & a_{5,4}(0) \\
a_{1,5}(0) & a_{2,5}(0) & a_{3,5}(0) & a_{4,5}(0) & a_{5,5}(0)
\end{bmatrix} = \\
\begin{bmatrix}
0.0093 \times 10^{-3} & -0.2211 \times 10^{-4} & -0.2109 & -0.0033 & -0.0014 \\
0.1326 \times 10^{-3} & 0.1117 \times 10^{-4} & -0.8945 & -0.0466 & -0.0208 \\
-0.1419 \times 10^{-3} & 0.1094 \times 10^{-4} & -1.1307 & 0.0500 & 0.0223
\end{bmatrix}.
\]

Also, for \( w = h = 0.000001 \), the values of \( a_{i,j}(h) \) are the same as that when \( w = 0 \) to the first few 
significant digits. Hence we do not present that here.

Now using \( a_{i,j}(w), S_i(w) \) and \( \phi_i(w) \) values for \( i = 1, 2, 3, 4, 5 \) and \( j = 3, 4, 5 \) at \( w = 0 \) and \( w = h \) in 
Equation (1) we can compute \( \tilde{H}_{ij}(x,w) \). In particular for \( x = 20 \) (which is what we need here) we get

\[
\begin{bmatrix}
\tilde{H}_{13}(20, 0) & \tilde{H}_{14}(20, 0) & \tilde{H}_{15}(20, 0) \\
\tilde{H}_{23}(20, 0) & \tilde{H}_{24}(20, 0) & \tilde{H}_{25}(20, 0) \\
\tilde{H}_{33}(20, 0) & \tilde{H}_{34}(20, 0) & \tilde{H}_{35}(20, 0) \\
\tilde{H}_{43}(20, 0) & \tilde{H}_{44}(20, 0) & \tilde{H}_{45}(20, 0) \\
\tilde{H}_{53}(20, 0) & \tilde{H}_{54}(20, 0) & \tilde{H}_{55}(20, 0)
\end{bmatrix} = \\
\begin{bmatrix}
0.1583 & 0.3995 & 0.4422 \\
0.1497 & 0.3913 & 0.4590 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The values of \( \tilde{H}_{ij}(20, h) \) for some small \( h \) do not differ in the first four significant digits from 
the corresponding \( \tilde{H}_{ij}(20, 0) \) values for \( i = 1, 2, 3, 4, 5 \) and \( j = 3, 4, 5 \), hence they are not reported.

2
Thus we have the expected number of days from $t = 0$ (with initial water level $X(0) = 20$ as well as initial environmental condition $Z(0) = 1$) for the water level to become nominal as

$$E[T] = (-1) \frac{d}{dw} \sum_{j=3}^{5} \tilde{H}_{1j}(20, w)|_{w=0} = - \lim_{h \to 0} \frac{\sum_{j=3}^{5} \tilde{H}_{1j}(20, h) - \tilde{H}_{1j}(20, 0)}{h}$$

which is approximately **15.593** days by using $h = 0.000001$. ■