Sensitivity Analysis and Duality
Part II – Duality

Based on
Chapter 6
Introduction to Mathematical Programming: Operations Research, Volume 1
4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

Lewis Ntaimo
6.8 – Duality

Associated with any LP is another LP, called the dual. Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear programming and nonlinear programming.

- Provides interesting economic and sensitivity analysis insights.

In this lecture you will learn:
- Finding the dual of an LP
- Economic interpretation of the dual LP
- Some basic duality theory
- Complementary Slackness
6.8 – Finding the Dual of an LP

When taking the dual of any LP, the given LP is referred to as the primal.

If the primal is a max problem, the dual will be a min problem and visa versa.
6.8 – Finding the Dual of an LP

Let us define (arbitrarily) the variables for a max problem to be 

\[ z, x_1, x_2, \ldots, x_n \]

and the variables for a min problem to be 

\[ w, y_1, y_2, \ldots, y_n. \]

A max problem in which all the variables are required to be nonnegative and all the constraints are \( \leq \) constraints is called a normal max problem. 

Similarly, a min problem in which all the variables are required to be nonnegative and all the constraints are \( \geq \) constraints is called a normal min problem.
6.8 – Finding the Dual of a normal LP

Normal max problem

\[
\max z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
s.t. \quad \begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq b_2 \\
\vdots & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m \\
x_j & \geq 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]

Normal min problem

\[
\min w = b_1y_1 + b_2y_2 + \ldots + b_my_m \\
s.t. \quad \begin{align*}
a_{11}y_1 + a_{12}y_2 + \ldots + a_{m1}y_m & \geq c_1 \\
a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m & \geq c_2 \\
\vdots & \quad \vdots \\
a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m & \geq c_n \\
y_i & \geq 0 \quad (i = 1, 2, \ldots, m)
\end{align*}
\]
6.8 – Finding the Dual of an LP (Compact Form)

**Normal max problem**

Primal

\[
\begin{align*}
\text{max } z &= cx \\
\text{s.t. } Ax &\leq b \\
x &\geq 0
\end{align*}
\]

Dual

\[
\begin{align*}
c: &\ 1 \times n \text{ row vector} \\
b: &\ m \times 1 \text{ column vector} \\
A: &\ m \times n \text{ matrix}
\end{align*}
\]

**Normal min problem**

Dual

\[
\begin{align*}
\text{min } w &= b^T y \\
\text{s.t. } A^T y &\geq c^T \\
y &\geq 0
\end{align*}
\]

Primal
6.8 – Economic Interpretation of the Dual Problem

Interpreting the Dual of the Dakota Problem

Primal:
Max z = \(60x_1 + 30x_2 + 20x_3\)
\[\begin{align*}
8x_1 + 6x_2 + x_3 &\leq 48 \quad \text{(Lumber constraint)} \\
4x_1 + 2x_2 + 1.5x_3 &\leq 20 \quad \text{(Finishing constraint)} \\
2x_1 + 1.5x_2 + 0.5x_3 &\leq 8 \quad \text{(Carpentry constraint)} \\
x_1, x_2, x_3 &\geq 0
\end{align*}\]

x_1 = \# of desks manufactured
x_2 = \# of tables manufactured
x_3 = \# of chairs manufactured

The dual is:
Min w = \(48y_1 + 20y_2 + 8y_3\)
\[\begin{align*}
8y_1 + 4y_2 + 2y_3 &\geq 60 \quad \text{(Desk constraint)} \\
6y_1 + 2y_2 + 1.5y_3 &\geq 30 \quad \text{(Table constraint)} \\
y_1 + 1.5y_2 + 0.5y_3 &\geq 20 \quad \text{(Chair constraint)} \\
y_1, y_2, y_3 &\geq 0
\end{align*}\]
6.9 – Economic Interpretation of the Dual Problem

Primal:

\[
\begin{align*}
\text{Max } z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t. } &8x_1 + 6x_2 + x_3 \leq 48 \quad \text{(Lumber constraint)} \\
&4x_1 + 2x_2 + 1.5x_3 \leq 20 \quad \text{(Finishing constraint)} \\
&2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \quad \text{(Carpentry constraint)} \\
&x_1, x_2, x_3 \geq 0
\end{align*}
\]

Dual:

\[
\begin{align*}
\text{Min } w &= 48y_1 + 20y_2 + 8y_3 \\
\text{s.t. } &8y_1 + 4y_2 + 2y_3 \geq 60 \quad \text{(Desk constraint)} \\
&6y_1 + 2y_2 + 1.5y_3 \geq 30 \quad \text{(Table constraint)} \\
&y_1 + 1.5y_2 + 0.5y_3 \geq 20 \quad \text{(Chair constraint)} \\
y_1, y_2, y_3 \geq 0
\end{align*}
\]

Relevant information about the Dakota problem dual is:

<table>
<thead>
<tr>
<th>Resource</th>
<th>Desk</th>
<th>Table</th>
<th>Chair</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber</td>
<td>8 board ft</td>
<td>6 board ft</td>
<td>1 board ft</td>
<td>48 boards ft</td>
</tr>
<tr>
<td>Finishing</td>
<td>4 hours</td>
<td>2 hours</td>
<td>1.5 hours</td>
<td>20 hours</td>
</tr>
<tr>
<td>Carpentry</td>
<td>2 hours</td>
<td>1.5 hours</td>
<td>0.5 hours</td>
<td>8 hours</td>
</tr>
<tr>
<td>Selling Price</td>
<td>$60</td>
<td>$30</td>
<td>$20</td>
<td></td>
</tr>
</tbody>
</table>
6.9 – Economic Interpretation of the Dual Problem

**Primal:**

\[
\begin{align*}
\text{max } z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad 8x_1 + 6x_2 + x_3 &\leq 48 \quad \text{(Lumber constraint)} \\
4x_1 + 2x_2 + 1.5x_3 &\leq 20 \quad \text{(Finishing constraint)} \\
2x_1 + 1.5x_2 + 0.5x_3 &\leq 8 \quad \text{(Carpentry constraint)} \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]

**Dual:**

\[
\begin{align*}
\text{min } w &= 48y_1 + 20y_2 + 8y_3 \\
\text{s.t.} \quad 8y_1 + 4y_2 + 2y_3 &\geq 60 \quad \text{(Desk constraint)} \\
6y_1 + 2y_2 + 1.5y_3 &\geq 30 \quad \text{(Table constraint)} \\
y_1 + 1.5y_2 + 0.5y_3 &\geq 20 \quad \text{(Chair constraint)} \\
y_1, y_2, y_3 &\geq 0
\end{align*}
\]

The first dual constraint is associated with **desks**, the second with **tables**, and the third with **chairs**. Decision variable \(y_1\) is associated with **lumber**, \(y_2\) with **finishing** hours, and \(y_3\) with **carpentry** hours.
6.9 – Economic Interpretation of the Dual Problem

Suppose an entrepreneur wants to purchase all of Dakota’s resources. The entrepreneur must determine the price he or she is willing to pay for a unit of each of Dakota’s resources.

To determine these prices we define:

\[ y_1 = \text{price paid for 1 boards ft of lumber} \]
\[ y_2 = \text{price paid for 1 finishing hour} \]
\[ y_3 = \text{price paid for 1 carpentry hour} \]

Actually, the resource prices \( y_1, y_2, \) and \( y_3 \) should be determined by simply solving the Dakota dual problem!
6.9 – Economic Interpretation of the Dual Problem

The total price that should be paid for these resources is

\[ 48y_1 + 20y_2 + 8y_3. \]

Since the cost of purchasing the resources is to minimize:

\[ \text{min } w = 48y_1 + 20y_2 + 8y_3 \]

is the **objective function** for the Dakota dual.

In setting **resource prices**, the prices must be high enough to induce Dakota to sell. For example, the entrepreneur must offer Dakota at least $60 for a combination of resources that includes 8 board feet of lumber, 4 finishing hours, and 2 carpentry hours because Dakota could, if it wished, use the resources to produce a desk that could be sold for $60. Since the entrepreneur is offering \( 8y_1 + 4y_2 + 2y_3 \) for the resources used to produce a desk, he or she must chose \( y_1, y_2, \) and \( y_3 \) to satisfy:

\[ 8y_1 + 4y_2 + 2y_3 \geq 60 \]
Similar reasoning shows that at least $30 must be paid for the resources used to produce a table. Thus $y_1$, $y_2$, and $y_3$ must satisfy:

$$6y_1 + 2y_2 + 1.5y_3 \geq 30$$

Likewise, at least $20 must be paid for the combination of resources used to produce one chair. Thus $y_1$, $y_2$, and $y_3$ must satisfy:

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$

The solution to the Dakota dual yields *prices* for lumber, finishing hours, and carpentry hours.

In summary, when the primal is a *normal max problem*, the dual variables are related to the *value of resources* available to the decision maker. For this reason, dual variables are often referred to as *resource shadow prices* (also, *dual multipliers*, *simplex multipliers*).
6.10 – Finding the Dual of a Nonnormal LP

Normal max problem
\[
\text{max } z = cx \\
\text{s.t. } Ax \leq b \\
\quad x \geq 0
\]

Normal min problem
\[
\text{min } w = b^T y \\
\text{s.t. } A^T y \geq c^T \\
\quad y \geq 0
\]

Many LPs are not normal max or min problems. To place a max problem into normal form, proceed as follows:

**Step 1.** Multiply each \( \geq \) constraint by -1, converting it into a \( \leq \) constraint.

**Step 2.** Replace each equality constraint by two inequality constraints
\( (a \leq \text{constraint and } a \geq \text{constraint}) \).

**Step 3.** Replace each urs variable \( x_i \) by \( x_i = x'_i - x''_i \), where \( x'_i, x''_i \geq 0 \).

To place a min problem into normal form, proceed as above but instead derive \( \geq \) constraints for the \( \leq \) and \( = \) constraints.
6.10 – Finding the Dual of a Nonnormal LP

Example 1. Nonnormal max problem

\[
\begin{align*}
\text{max } z &= 2x_1 + x_2 \\
\text{s.t. } & \quad x_1 + x_2 = 2 \\
& \quad 2x_1 - x_2 \geq 3 \\
& \quad x_1 - x_2 \leq 1 \\
& \quad x_1 \geq 0, x_2 \text{ urs.}
\end{align*}
\]

Normal max problem (primal)

\[
\begin{align*}
\text{max } z &= 2x_1 + x_2' - x_2'' \\
\text{s.t. } & \quad x_1 + x_2' - x_2'' \leq 2 \\
& \quad -x_1 - x_2' \leq -2 \\
& \quad -2x_1 + x_2' \leq -3 \\
& \quad x_1 - x_2' \leq 1 \\
& \quad x_1, x_2', x_2'' \geq 0.
\end{align*}
\]

Normal min problem (dual)

\[
\begin{align*}
\text{min } w &= 2y_1 - 2y_2 - 3y_3 + y_4 \\
\text{s.t. } & \quad y_1 - y_2 - 2y_3 + y_4 \geq 2 \\
& \quad y_1 - y_2 + y_3 - y_4 \geq 1 \\
& \quad -y_1 + y_2 - y_3 + y_4 \geq -1 \\
& \quad y_1, y_2, y_3, y_4 \geq 0.
\end{align*}
\]

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6.10 – Finding the Dual of a Nonnormal Max LP Directly

To place a max problem into normal form, proceed as follows:

**Step 1.** If the \( i \)th primal constraint is a \( \geq \) constraint, then the corresponding dual variable \( y_i \) must satisfy \( y_i \leq 0 \).

**Step 2.** If the \( i \)th primal constraint is a \( = \) constraint, then the dual variable \( y_i \) is urs.

**Step 3.** If the \( i \)th primal variable \( x_i \) is urs, then the \( i \)th dual constraint will be an \( = \) constraint.
6.10 – Finding the Dual of a Nonnormal Min LP Directly

To place a min problem into normal form, proceed as follows:

**Step 1.** If the ith primal constraint is a \( \leq \) constraint, then the corresponding dual variable \( x_i \) must satisfy \( x_i \leq 0 \).

**Step 2.** If the ith primal constraint is a \( = \) constraint, then the dual variable \( x_i \) is urs.

**Step 3.** If the ith primal variable \( y_i \) is urs, then the ith dual constraint will be an \( = \) constraint.
6.10 – Finding the Dual of a Nonnormal LP

Example 1. Nonnormal max problem

\[
\begin{align*}
\text{max } z &= 2x_1 + x_2 \\
\text{s.t. } &x_1 + x_2 = 2 \\
&2x_1 - x_2 \geq 3 \\
&x_1 - x_2 \leq 1 \\
&x_1 \geq 0, \text{ } x_2 \text{ ur.}
\end{align*}
\]

Normal min problem (dual)

\[
\begin{align*}
\min w &= 2y_1 + 3y_2 + y_3 \\
\text{s.t. } &y_1 + 2y_2 + y_3 \geq 2 \\
&y_1 - y_2 - y_3 = 1 \\
&y_1 \text{ ur.}, \text{ } y_2 \leq 0, \text{ } y_3 \geq 0.
\end{align*}
\]

Normal max problem (primal)

\[
\begin{align*}
\text{max } z &= 2x_1 + x_2' - x_2'' \\
\text{s.t. } &x_1 + x_2' - x_2'' \leq 2 \\
&-x_1 - x_2' + x_2'' \leq -2 \\
&-2x_1 + x_2' - x_2'' \leq -3 \\
&x_1 - x_2' + x_2'' \leq 1 \\
&x_1, x_2', x_2'' \geq 0.
\end{align*}
\]

Normal min problem (dual)

\[
\begin{align*}
\min w &= 2y_1 - 2y_2 - 3y_3 + y_4 \\
\text{s.t. } &y_1 - y_2 - 2y_3 + y_4 \geq 2 \\
&y_1 - y_2 + y_3 - y_4 \geq 1 \\
&-y_1 + y_2 - y_3 + y_4 \geq -1 \\
&y_1, y_2, y_3, y_4 \geq 0.
\end{align*}
\]
### 6.10 – Relation Between Primal and Dual Variables and Constraints

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Maximize</strong></td>
<td><strong>Minimize</strong></td>
</tr>
<tr>
<td><strong>Variables</strong></td>
<td><strong>Constraints</strong></td>
</tr>
<tr>
<td>$\geq 0$</td>
<td>$\leq c_i$</td>
</tr>
<tr>
<td>$\leq 0$</td>
<td>$\geq c_i$</td>
</tr>
<tr>
<td>urs</td>
<td>$= c_i$</td>
</tr>
<tr>
<td><strong>Constraints</strong></td>
<td><strong>Variables</strong></td>
</tr>
<tr>
<td>$\leq b_i$</td>
<td>$\geq 0$</td>
</tr>
<tr>
<td>$\geq b_i$</td>
<td>$\leq 0$</td>
</tr>
<tr>
<td>$= b_i$</td>
<td>urs</td>
</tr>
</tbody>
</table>
6.11 – Duality Theory

Let c: (1 x n) row vector
b: (m x 1) column vector
A: (m x n) matrix with m linearly independent rows.
x: (n x 1) vector of decision variables

Consider the following primal problem and its dual:

Primal (P)

\[
\text{max } z = cx \\
\text{s.t. } Ax \leq b \\
x \geq 0
\]

Dual (D)

\[
\text{min } w = b^T y \\
\text{s.t. } A^T y \geq c^T \\
y \geq 0
\]
6.11 – Duality Theory

Weak Duality (Theorem 1):
If x is a feasible solution to the primal (max) problem and y is a feasible solution to the dual problem, then
\[ cx \leq b^T y. \]

Proof:
Because \( y \geq 0 \), multiplying the primal constraints in (P) by \( y \) yields the following:
\[ y^T Ax \leq b^T y \quad (1) \]
Because \( x \geq 0 \), multiplying the dual constraints in (D) by \( x \) yields the following:
\[ x^T A^T y \geq c^T x \Rightarrow y^T Ax \geq c^T x. \quad (2) \]
Combining (1) and (2), we obtain
\[ c^T x \leq y^T Ax \leq b^T y \]
which is the desired result.
6.11 – Duality Theory (Basics)

Corollary 2:

(a) If the primal is unbounded, then the dual is infeasible.

(b) If the dual is unbounded, then the primal is infeasible.

Proof:

Suppose that the optimal objective value in the primal is $\infty$ and that the dual problem has a feasible solution $y$. By weak duality, $y$ satisfies $cx \leq b^Ty$. Taking the maximum over all primal feasible $x$, we conclude that $\infty \leq b^Ty$, which is impossible. This shows that the dual cannot have a feasible solution, thus establishing part (a). Part (b) follows by a symmetrical argument.

- **P:** max $z = cx$
  s.t. $Ax \leq b$
  $x \geq 0$

- **D:** min $w = b^Ty$
  s.t. $A^Ty \geq c^T$
  $y \geq 0$
Corollary 3:
Let \( x' \) and \( y' \) be feasible solutions to the primal and the dual, respectively, and suppose that \( cx' = b^T y' \).
Then \( x' \) and \( y' \) are optimal solutions to the primal and the dual, respectively.

Proof:
Let \( x' \) and \( y' \) be as given in the corollary. For every primal feasible solution \( x \), the weak duality theorem yields \( cx \leq b^T y' \). Thus since \( cx' = b^T y' \) this proves that \( x' \) is optimal for the primal.

Now consider every dual feasible solution \( y \), the weak duality theorem yields \( cx' \leq b^T y \). Thus since \( cx' = b^T y' \) this proves that \( y' \) is optimal for the dual.
6.11 – Duality Theory

**Strong Duality (Theorem 4):**
If an LP has an optimal solution, so does its dual, and the respective optimal objective function values are equal.

**Proof (outline):**
- Consider the standard form problem shown on the right.
- The independence assumption on \( A \) still holds.
- Apply the simplex method to this problem (avoid cycling by using an anti-cycling rule, e.g. Bland’s rule)
- The simplex method terminates with an optimal solution \( x \) and an optimal basis \( B \). Let \( x_{BV} = B^{-1}b \) be the corresponding vector of basic variables.
- When the simplex method terminates, the reduced costs must be nonnegative: \( c_{BV}B^{-1}A - c^T \geq 0 \).
- Define \( y = c_{BV}B^{-1} \). We then have \( A^Ty \geq c^T \), which shows that \( y \) is a feasible solution to the dual, which is shown on the right.
- In addition, \( y^Tb = c_{BV}B^{-1}b = c_Bx_{BV} = cx \).
- It follows that \( y \) is an optimal solution to the dual (See Corollary 3), and the optimal dual objective value is equal to the optimal primal objective value.
6.11 – Duality Theory

Assignment

How to read the optimal dual solution from row 0 of the optimal tableau:

Read pages 310 - 312 of Winston!
6.12 – Duality and Sensitivity Analysis

The proof of the strong duality theorem demonstrated the following result (max problem):

Assuming that a set of basic variables $BV$ is feasible, then $BV$ is optimal (that is, each variable in row 0 has a non-negative coefficient) if and only if the associated dual solution ($y = c_{BV}B^{-1}$) is dual feasible.

The result can be used for an alternative way of doing the following types of sensitivity analysis:

1. Changing the objective function coefficient of a nonbasic variable.
4. Changing the column of a nonbasic variable.
5. Adding a new activity.

In each case, the change leaves $BV$ feasible. $BV$ will remain optimal if the $BV$ row 0 remains non-negative.
6.12 – Duality and Sensitivity Analysis

Primal optimality and dual feasibility are equivalent:

The three changes listed in the previous slide will leave the current basis optimal if and only if the current dual solution \( y = c_{BV}B^{-1} \) remains dual feasible. If the current dual solution is no longer dual feasible, then BV will be suboptimal, and a new optimal solution must be found.
6.12 – Duality and Sensitivity Analysis

Example: Recall the Dakota problem

Max \( z = 60x_1 + 30x_2 + 20x_3 \)

s.t. \( 8x_1 + 6x_2 + x_3 \leq 48 \) (Lumber constraint)

\( 4x_1 + 2x_2 + 1.5x_3 \leq 20 \) (Finishing constraint)

\( 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \) (Carpentry constraint)

\( x_1, x_2, x_3 \geq 0 \)

The optimal solution is \( z = 280, s_1 = 24, x_3 = 8, x_1 = 2, x_2 = 0, s_2 = 0, s_3 = 0 \). The only nonbasic decision in the optimal solution is \( x_2 \) (tables). The dual of the Dakota problem is:

Min \( w = 48y_1 + 20y_2 + 8y_3 \)

s.t. \( 8y_1 + 4y_2 + 2y_3 \geq 60 \) (Desk constraint)

\( 6y_1 + 2y_2 + 1.5y_3 \geq 30 \) (Table constraint)

\( y_1 + 1.5y_2 + 0.5y_3 \geq 20 \) (Chair constraint)

\( y_1, y_2, y_3 \geq 0 \)

The optimal dual solution is \( w = 280, y_1 = 0, y_2 = 10, y_3 = 10 \). Let us now see how the knowledge of duality can be applied to sensitivity analysis.
6.12 – Duality and Sensitivity Analysis

1. Changing the Objective Function Coefficient of a Nonbasic Variable.

Let $c_2$ be the objective function coefficient of $x_2$ (tables), which is a nonbasic variable. Note that $c_2$ is the price at which a table is sold. For what values of $c_2$ will the current basis remain optimal?

If $y_1 = 0, y_2 = 10, y_3 = 10$ remains dual feasible, then the current basis (and the values of all the variables) are unchanged. Note that a change in $c_2$ will only affect the second dual constraint (table constraint):

\[ 6y_1 + 2y_2 + 1.5y_3 \geq c_2 \]

The first and third dual constraints will remain unchanged.

If $y_1 = 0, y_2 = 10, y_3 = 10$ satisfies the above constraint, then dual feasibility (and therefore, primal optimality) is maintained. Thus the current basis remains optimal if $c_2$ satisfies:

\[ 6(0) + 2(10) + 1.5(10) \geq c_2 \quad \text{or} \quad c_2 \leq 35. \]

This agrees with the result we obtained using “formal” sensitivity analysis.
6.12 – Duality and Sensitivity Analysis

1. Changing the Objective Function Coefficient of a Nonbasic Variable
   Cont...

   Using shadow (dual) prices, we may give an alternative interpretation of
   the previous result:

   Using shadow (dual) prices a table uses $6(0) + 2(10) + 1.5(10) =
   $35 worth of resources. So the only way producing a table can
   increase Dakota’s revenues is if a table sells for more than $35.
   Thus the current basis fails to be optimal if $c_2 > 35$, and the
   current basis remains optimal is $c_2 \leq 35$. 

6.12 – Duality and Sensitivity Analysis


Suppose the table sells for $43 and uses 5 board feet of lumber, 2 finishing hours, and 2 carpentry hours. Does the current basis remain optimal?

Changing the column for the nonbasic variable “tables” leaves the first and third constraints unchanged but changes the second constraint to

\[ 5y_1 + 2y_2 + 2y_3 \geq 43 \]

Because \( y_1 = 0, y_2 = 10, y_3 = 10 \) does not satisfy the new dual constraint, then dual feasibility (and therefore, primal optimality) is not maintained. Thus the current basis is no longer optimal.

In terms of shadow (dual) prices each table uses $40 worth of resources and sells for $43, so Dakota can increase its revenue by 43 - 40 = 3 for each table that is produced. Thus the current basis is no longer optimal, and \( x_2 \) (tables) will be basic in the new optimal solution.
6.12 – Duality and Sensitivity Analysis

5. Adding a New Activity.

Suppose Dakota is considering manufacturing footstools ($x_4$). A footstool sells for $15 and uses 1 board foot of lumber, 1 finishing hour, and 1 carpentry hour. Does the current basis remain optimal?

Introducing the new activity (footstools) leaves the three dual constraints unchanged but the new variable $x_4$ adds a new dual constraint

$$y_1 + y_2 + y_3 \geq 15$$

The current basis remains optimal if $y_1 = 0$, $y_2 = 10$, $y_3 = 10$ satisfies the new dual constraint. Because $0 + 10 + 10 \geq 15$ the current basis remains optimal.

In terms of shadow (dual) prices a stool uses $0 + 10 + 10 = $20 worth of resources and sells for only $15, so Dakota should not make footstools, and the current basis remains optimal.
6.13 – Complementary Slackness

The **Theorem of Complementary Slackness** is an important result that relates the optimal primal and dual solutions. To state this theorem, we assume that the primal is a normal max problem with variables $x_1, x_2, \ldots, x_n$ and $m \leq$ constraints. Let $s_1, s_2, \ldots, s_m$ be the slack variables for the primal. Then the dual is a normal min problem with variables $y_1, y_2, \ldots, y_m$ and $n \geq$ constraints. Let $e_1, e_2, \ldots, e_n$ be the slack variables for the dual.

Max $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. \[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 = b_2 \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_m \]
\[ x_j \geq 0 \ (j = 1, 2, \ldots, n), \ s_i \geq 0 \ (i = 1, 2, \ldots, m). \]

Min $w = b_1y_1 + b_2y_2 + \ldots + b_my_m$

s.t. \[ a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m - e_1 = c_1 \]
\[ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m - e_2 = c_2 \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m - e_n = c_n \]
\[ y_i \geq 0 \ (i = 1, 2, \ldots, m), \ e_j \geq 0 \ (j = 1, 2, \ldots, n). \]
6.13 – Complementary Slackness

Theorem of Complementary Slackness

Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) be a feasible primal solution and \( y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \) be a feasible dual solution.

Then \( x \) is primal optimal and \( y \) is dual optimal if and only if

\[
\begin{align*}
  s_i y_i &= 0 \quad (i = 1, 2, \ldots, m) \quad (1) \\
  e_j x_j &= 0 \quad (i = 1, 2, \ldots, m). \quad (2)
\end{align*}
\]

From (1) it follows that the optimal and dual solutions must satisfy:

\[
\begin{align*}
  \text{ith primal slack} > 0 & \implies \text{ith dual variable} = 0 \quad (3) \\
  \text{ith dual variable} > 0 & \implies \text{ith primal slack} = 0 \quad (4)
\end{align*}
\]

From (2) it follows that the optimal and dual solutions must satisfy:

\[
\begin{align*}
  \text{ith dual excess} > 0 & \implies \text{jth primal variable} = 0 \quad (5) \\
  \text{jth primal variable} > 0 & \implies \text{jth dual excess} = 0 \quad (6)
\end{align*}
\]

From (3) and (5), we see that if a constraint in either the primal or dual is non-binding (has either \( s_1 > 0 \) or \( e_j > 0 \)), then the corresponding variable in the other (or complementary) problem must be zero. Hence the name \textbf{complementary slackness}.
6.13 – Complementary Slackness

Example: Use complementary slackness to solve the Dakota problem.

Max \( z = 60x_1 + 30x_2 + 20x_3 \)

s.t. \( 8x_1 + 6x_2 + x_3 \leq 48 \) (Lumber constraint)
      \( 4x_1 + 2x_2 + 1.5x_3 \leq 20 \) (Finishing constraint)
      \( 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \) (Carpentry constraint)

\( x_1, x_2, x_3 \geq 0 \)

The optimal solution is \( z = 280, s_1 = 24, x_3 = 8, x_1 = 2, x_2 = 0, s_2 = 0, s_3 = 0 \) and the dual of the Dakota problem is:

Min \( w = 48y_1 + 20y_2 + 8y_3 \)

s.t. \( 8y_1 + 4y_2 + 2y_3 \geq 60 \) (Desk constraint)
      \( 6y_1 + 2y_2 + 1.5y_3 \geq 30 \) (Table constraint)
      \( y_1 + 1.5y_2 + 0.5y_3 \geq 20 \) (Chair constraint)

\( y_1, y_2, y_3 \geq 0 \)
Example: Use complementary slackness to solve the Dakota problem.

Since optimal primal solution is \( z = 280 \), \( s_1 = 24 \), \( x_3 = 8 \), \( x_1 = 2 \), \( x_2 = 0 \), \( s_2 = 0 \), \( s_3 = 0 \) we have:

\[
\begin{align*}
    s_1 &= 48 - (8(2) + 6(0) + 1(8)) = 24 \\
    s_2 &= 20 - (4(2) + 2(0) + 1.5(8)) = 0 \\
    s_3 &= 8 - (2(2) + 1.5(0) + 0.5(8)) = 0
\end{align*}
\]

From complementary slackness we have that:

\[
\begin{align*}
    s_1y_1 &= s_2y_2 = s_3y_3 = 0 \\
    e_1x_1 &= e_2x_2 = e_3x_3 = 0
\end{align*}
\]

Because \( s_1 > 0 \) condition (3) implies that \( y_1 = 0 \): \( s_1y_1 = 0 \) we have that \( 24y_1 = 0 \Rightarrow y_1 = 0 \).

Because \( x_1 > 0 \) and \( x_3 > 0 \) condition (6) implies that the first and third dual constraints must be binding. Thus the optimal values of \( y_2 \) and \( y_3 \) must satisfy:

\[
\begin{align*}
    8y_1 + 4y_2 + 2y_3 &= 60 \\
    y_1 + 1.5y_2 + 0.5y_3 &= 20
\end{align*}
\]

Solving simultaneously yields: \( y_2 = 10 \) and \( y_3 = 10 \). Therefore, the optimal dual objective value is \( w = 48(0) + 20(10) + 8(10) = 280 \). From Strong Duality, of course, we know that \( w = z = 280 \).